PHYSICAL SOCIETY OF LONDON

MEMOIRS

OF THE

COLLEGE OF SCIENCE

KYOTO IMPERIAL UNIVERSITY

VOL. V, No. 4

FEBRUARY 1922

KYOTO

On the Octahydrate of Magnesium Sulphate

By

Shirow Takegami

(Received July 4, 1921)

Of magnesium sulphate, there are various hydrates. Besides the well known hydrates with 1, 4, 5, 6, 7, and 12 molecules of the water of crystallisation, the hydrates with 5/4, 2, and 3 molecules of water also exist.

Among these hydrates, the hydrates with 1, 6, 7, and 12 molecules of water are stable in contact with the pure solution of magnesium sulphate each within certain limits of temperature, but the others are obtained chiefly from a solution containing magnesium chloride in addition to it. The hexa- and hepta- hydrates exist each in two modifications.

The transition temperatures between two successive ones among the four kinds of the hydrates which are stable in contact with the pure solution, are as follows:

$$\text{Dodeca.} \overset{1.8^\circ}{\rightleftarrows} \text{Hepta.} \overset{42.2^\circ}{\rightleftarrows} \text{Hexa.} \overset{68.0^\circ}{\rightleftarrows} \text{Mono.}$$

This is a general outline of the results[1] obtained by the previous investigators.

On the author's previous investigation[2] of the reciprocal salt pairs: $Na_2Cl_2 + MgSO_4 \rightleftarrows Na_2SO_4 + MgCl_2$ at 25°C., there was one special

---

1 Abegg: Handbuch der anorg. Chem. II. **2**, p. 56; Gmelin-Kraut-Friedheim: Handbuch der anorg. Chem. II. **2**, p. 405; Seidell: Solubilities of Inorganic and Organic Compounds. 2d. ed., p. 396.

2 Journ. Tokyo Chem. Soc., **41**, 831 (1920); These Memoirs, **4**, 317 (1921).

case in the system of magnesium sulphate and sodium sulphate, in which the magnesium sulphate as the residue formed the octahydrate. In the literature, however, there was no mention of the octahydrate of magnesium sulphate, and moreover, in the other two cases the residue was the heptahydrate, it was attributed to an experimental error, and the author arrived at the conclusion that the magnesium sulphate which forms the residue in the system of magnesium sulphate and sodium sulphate at 25°C., was the heptahydrate.

But during the investigations which were made afterwards to ascertain whether the above conclusion were true or not, the existence of the octahydrate of magnesium sulphate became very probable. Therefore, on the suggestion of Prof. Y. Osaka, the study of the equilibrium of the system of magnesium sulphate, sodium sulphate, and water was again undertaken.

The experiments were carried out in the same way as in the case of the reciprocal salt pairs. But, in order to have magnesium sulphate as the only residue, a very large excess of magnesium sulphate was used and to secure the complete equilibrium, the equilibrium flask was warmed previously to dissolve the salts completely, and then rotated in a thermostat.

I. *The system of magnesium sulphate, sodium sulphate, and water at 25°C.*

The results are represented in Table 1, 2, and Fig. 1.

TABLE 1.

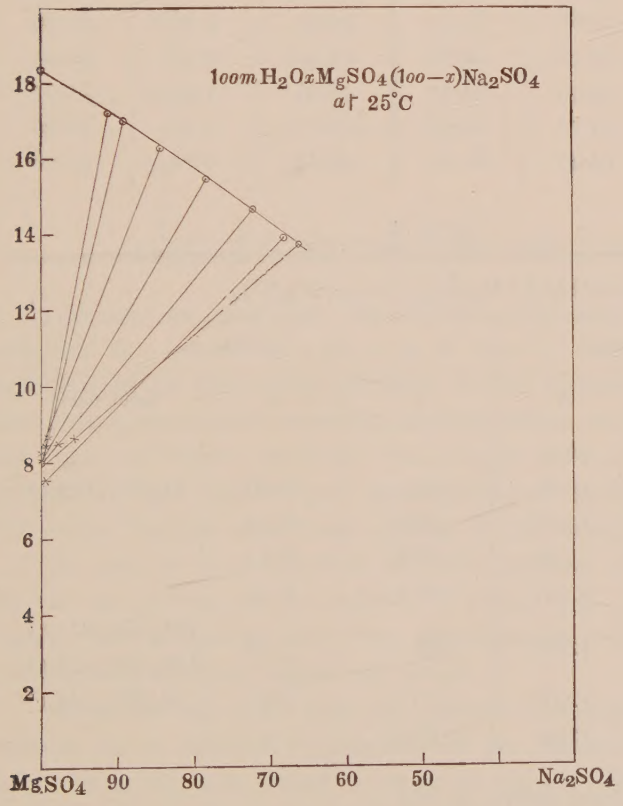
No.	Composition of solution in gram percentages.			Composition of residue in gram percentages.		
	$MgSO_4$	Na_2SO_4	H_2O	$MgSO_4$	Na_2SO_4	H_2O
1.	26·68	.....	73·32	.....	.....	.....
2.	25·48	2·73	71·79	43·99	0·25	55·76
3.	25·04	3·54	71·42	43·01	0·53	56·46
4.	24·49	5·20	70·31	44·63	0·00	55·37
5.	23·49	7·50	69·01	45·17	0·15	54·68
6.	22·32	10·10	67·58	42·91	1·21	55·88
7.	21·79	11·90	66·31	46·69	0·30	53·01
8.	21·27	12·76	65·97	41·65	2·25	56·10

TABLE 2.

$100mH_2O.xMgSO_4(100\text{-}x)\,Na_2SO_4$

No.	Solution.		Residue.		
	x	m	x	m	Solid phase.
1.	100·00	18·31	.....	.....	$MgSO_4.8H_2O$
2.	91·30	17·20	99·50	8·44	,,
3.	89·30	17·03	98·95	8·69	,,
4.	84·75	16·28	100·00	8·30	,,
5.	78·70	15·47	99·75	808	,,
6.	72·27	14·64	97·67	8·51	,,
7.	68·36	13·92	99·46	7·55	,,
8.	66·30	13·75	95·60	8·61	,,

Fig. 1.



It may be clearly seen from Fig. 1 that whenever it was considered to have attained the complete equilibrium, the residue was always the octahydrate. The reason why the heptahydrate was obtained in the author's previous work, is perhaps due to the fact that the heptahydrate was shaken with the solution and was analysed before it reached equilibrium.

If the proportion of sodium sul-

phate was increased to some definite point, a transition point of the octahydrate to the heptahydrate might have been found as in the case at 30°C. in the next section, but the formation of astrakanite prevented the further experiment.

2. *The system of magnesium sulphate, sodium sulphate, and water at 30° C.*

The results are represented in Tables 3, 4, and Fig. 2.

TABLE 3.

No.	Composition of solution in gram percentages.			Composition of residue in gram percentages.		
	$MgSO_4$	Na_2SO_4	H_2O	$MgSO_4$	Na_2SO_4	H_2O
*	29·0	.....	71·0	.....	.....	.....
1.	26·35	3·96	69·69	42·28	0·79	56 93
2.	24·35	8·43	67·22	43·14	1·27	55·59
3.	22·89	11·80	65·31	42·95	1·76	55·29
4.	22·82	12·56	64·62	42·99	2·06	54·95
5.	23·00	12·49	64·51	45·91	1·04	53·05
6.	22·33	13·71	63·96	45·77	1·26	52·97
7.	21·92	14·47	63·61	46·58	0·84	52·58

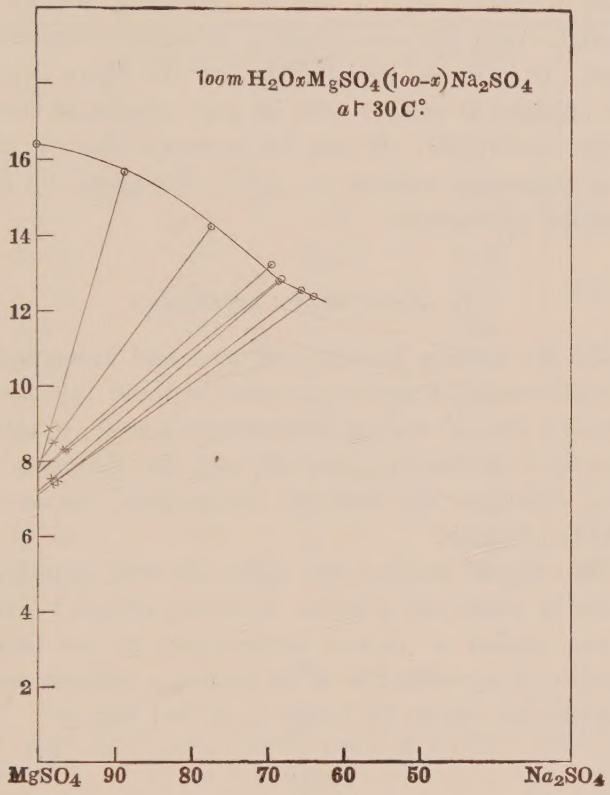
TABLE 4.

$100mH_2O.xMgSO_4\,(100\text{-}x)\,Na_2SO_4$

No.	Solution.		Residue.		
	x	m	x	m	Solid phase.
*	100·0	16·4	.....	.....	
1.	88·67	15·69	98·43	8·86	$MgSO_4.8H_2O$
2.	77·33	14·28	98·05	8·45	,,
3.	69·60	13·28	96·64	8·32	,,
4.	68·19	12·91	96.11	8·22	,,
5.	68·49	12·85	98·11	7·58	$MgSO_4.8H_2O$; $MgSO_4.7H_2O$
6.	65·78	12·60	97·70	7·56	$MgSO_4.7H_2O$
7.	64·13	12·45	98·25	7·42	,,

* Determination by Mulder (Seidell, *loc. cit.*).

Fig. 2.



As may be seen from Fig. 2, when the molar fraction of sodium sulphate in a solution reaches 0·32 or more of the whole sulphate, the octahydrate of magnesium sulphate as the residue is transformed into the heptahydrate, and there is a marked difference in solubility between the octa- and hepta- hydrate. Accordingly it is evident that the octahydrate is more stable than the heptahydrate under the solution where the molar fraction of sodium sulphate in the solution is less than 0·32.

It must be here remarked that the residue, which was proved to be the octahydrate on its analysis in a state still wet with the mother liquor, was converted into the heptahydrate, when it was pressed between folded filter papers and dried.

What we can infer from the present investigation of the ternary system, is as follows: According to the literature, the crystals of magnesium sulphate which is stable in contact with its pure solution at 25°C. and 30°C., is the heptahydrate, but it has become clear from

the present investigation that at these temperatures, magnesium sulphate which exists in contact with a solution containing sodium sulphate is the octahydrate until the concentration of sodium sulphate reaches a certain limit. It may also be inferred from the figure that crystals of magnesium sulphate in contact with its pure solution at those temperatures are the octahydrate. It may be presumed that the data for the solubility of magnesium sulphate at 30°C., determined by Mulder, is a solubility of the octahydrate.

3. *Some further experiments*

To make the relation between the octa- and hepta- hydrates more clear, a measurement of vapour pressure of these salts and solutions was undertaken, but, as various difficulties occurred, it was given up. Then a transition cell was constructed, and the thermometer method was tried to determine the transition temperature, but no satisfactory result could be obtained.

Thus the original method was again adopted, namely, an equilibrium system in which the solution, containing always 7–9% of sodium sulphate, was studied at various temperatures to see their influence upon the water of crystallisation of the residue at different temperatures.

The results are shown in Tables 5, 6, and Fig. 3.

TABLE 5.

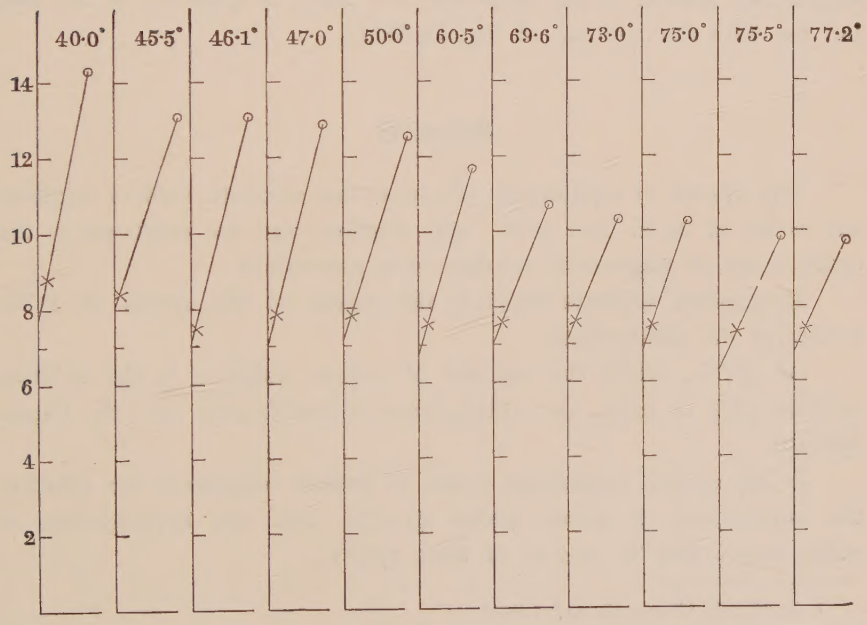
Temp.	Composition of solution in gram percentages.			Composition of residue in gram percentages.		
	$MgSO_4$	Na_2SO_4	H_2O	$MgSO_4$	Na_2SO_4	H_2O
40·0	29·58	2·47	67·95	42·57	0·61	56·82
45·5	30·96	3·24	65·80	43·97	0·40	55·63
46·1	31·22	2·97	65·81	46·39	0·33	52·78
47·0	31·56	2·88	65·56	45·42	0·55	54·03
50·0	31·68	3·40	64·92	45·60	0·62	50·78
60·5	33·69	3·03	63·28	46·30	0·68	53·02
69·6	35·48	3·34	61·18	46·24	0·65	53·11
73·0	36 40	3·14	60·46	46·29	0·55	53·16
75·0	36·92	2·77	60·31	46·45	0·67	52·88
75·5	36·81	3·98	59·21	46·32	1·39	52·29
77·2	37·53	3·39	59·08	46·60	0·87	52·53

TABLE 6.

$100mH_2O \cdot xMgSO_4 \cdot (100-x)\,Na_2SO_4$

Temp.	Solution.		Residue.		
	x	m	x	m	Solid phase.
40·0	93·37	14·35	98·79	8·82	$MgSO_4.8H_2O$
45·5	91·86	13·06	99·23	8·40	,,
46·1	92·54	13·05	99·42	7·49	$MgSO_4.7H_2O$
47·0	92·80	12·89	98·97	7·84	,,
50·0	91·67	12·57	98·85	7·80	,,
60·5	92·92	11·68	98·76	7·56	,,
69·6	92·61	10·68	98·82	7·59	,,
73·0	93·20	10·35	98·99	7·60	,,
75·0	94·01	10·27	98·79	7·52	,,
75·5	91·61	9 86	97·51	7·36	$MgSO_4\,6H_2O$(?)
77·2	92·91	9·78	98·46	7·42	$MgSO_4.7H_2O$

Fig. 3.



The hydrates of magnesium sulphate in contact with a solution containing 7–9% of sodium sulphate, have a transition point of the octa- to the hepta- hydrate at the temperature of 45·5°–46·1°C.

The system in which a hydrate of magnesium sulphate is the residue and is in equilibrium with a solution containing sodium sulphate in addition to magnesium sulphate, is a bivariant system, so the transition temperature of the octa- to the hepta- hydrate varies when the concentration of sodium sulphate increases. Therefore the temperature of 48·2°C. which is determined as the transition temperature of the hepta- to the hexa- hydrate by means of the dilatometer by van der Heide,[1] seems to correspond to that of the octa- to the hepta-hydrate in the present investigation. Next, it was naturally suspected that the temperature of 68°C. which is known as the transition temperature of the hexa- to the mono- hydrate, might correspond to the transition temperature of the hepta- to the hexa- hydrate in the present case, but no hexahydrate could be obtained up to 77·2°C., except one case at 75·5°C., where the residue was rather the hexa-hydrate. Accordingly within the limit of the present investigation, a special temperature corresponding to 68°C., could not be found.

As the existence of octahydrate of magnesium sulphate seems to be unquestionable, the number of solid phases which exist in the equilibrium system of the reciprocal salt pairs, in question, is not nine but ten with the addition of $MgSO_4.8H_2O$.

Summary

The system of equilibrium of magnesium sulphate, sodium sulphate, and water at 25°C. and 30°C. was studied; and the existence of the octahydrate of magnesium sulphate was ascertained.

Magnesium sulphate which is the residue in this system at 25°C. is always the octahydrate.

At 30°C., when the amount of sodium sulphate in the solution reaches 32% or more, the octahydrate is transformed into the hepta-hydrate.

In the system containing 7–9% of sodium sulphate in the solution, the octahydrate is stable under 45·5°C. and the heptahydrate is stable above 46·1°C. and to at least 77·2°C.

1 Z. physik. Chem., **12**, 417 (1893)

Lastly it may be concluded that the transition temperatures of the hydrates of magnesium sulphate as inferred from the case in which sodium sulphate coexists, as follows:

$$\text{Octa.} \underset{48{\cdot}2^\circ}{\rightleftharpoons} \text{Hepta.} \underset{\text{over } 77{\cdot}2^\circ}{\rightleftharpoons} \text{Hexa.}$$

In conclusion, the author wishes to express his sincere thanks to Prof. Y. Osaka for his valuable suggestions and kind instructions.

On the Figures produced on Photographic Plates by Electric Discharges

By

Shinsuke Tanaka

(Received July 4, 1921)

One of the poles of an influence machine is connected through a short spark gap to a metallic electrode placed on the sensitive film of a photographic plate, and the other pole is connected to a smooth metallic plate placed under the photographic plate. When a momentary brush discharge occurs in this case at the metallic electrode, an impression of a beautiful negative or positive discharge figure, according to the sign of the electrode, will be produced on the plate after development.

S. Mikola[1] observed that, if a small portion of the sensitive film of the photographic plate around the electrode was covered by water, a curious discharge figure was impressed on the plate. Two different brush discharges of opposite polarities started at the boundary of the water and extended in opposite directions, the brush which extended outward being of the same polarity as the electrode on the photographic plate.

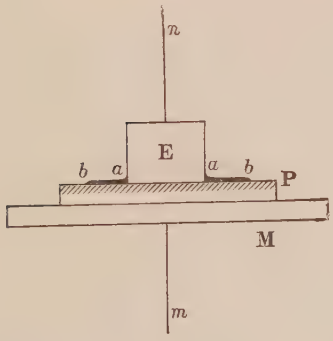


Fig. 1.

Mikola's experimental arrangement is illustrated, diagramatically in Fig. 1. Here E denotes the metallic electrode, M

1 S. Mikola, Phys. Z. S., 18, 158, (1917).

the smooth metallic plate, P the photographic plate and ab a thin layer of water. When E was positive the positive brush started at the boundary b of the water and extended outward, and the negative brush which started at the same boundary b extended in the opposite direction. Next when E was made negative the polarities of these two brushes became opposite respectively. Fig. 2 in the annexed photographs obtained by the writer is reproduced to show such a behavior of the brush discharges.

The writer has examined this behavior of the brush discharge using liquids having different electric conductivities, and gave a rough explanation of the formation of the figures. When a liquid of comparatively large electric conductivity, such as a dilute aqueous salt solution or a dilute aqueous acid solution, was taken to wet the photographic plate, no impression of the brush discharge was produced in the wet part of the photographic plate. The liquid behaved in this case as if it were a part of the electrode on the plate, and only the brush of the same sign as that of the electrode started at and extended outward from the boundary of the liquid. Next when a comparatively thick layer of the order of 2 mms. or more of the distilled water or of the water from the water pipes was employed to cover the plate, the appearance of the brush discharge was essentially the same as above as shown in Fig. 3. Actually the writer could ascertain by visual observation that, in these cases, the visible discharge took place only at the region outside the boundary of the liquid, and that no visible discharge occurred at the portion covered by the liquid. This was naturally expected, and the liquid behaved in this case, on account of their comparatively small electric resistance, as a part of the electrode on the photographic plate.

When a liquid of bad conductivity such as oils of various kinds or absolute alcohol was used instead of water, the brush discharge of the same polarity as the electrode started directly at the electrode and extended along the surface of the liquid and thence to the surface of the sensitive film of the photographic plate. Only the portion of the brush-impression just under the liquid, appearing somewhat weak and diffuse; whilst the main feature of the figure was entirely the same as the ordinary discharge figure on the photographic plate as shown in Fig. 4. The correctness of the above statement was verified directly by visual observations. The total electric resistance of the liquid with which the photographic plate is covered, could also be increased by

reducing its thickness. Thus the effect of greatly reducing the thickness of the water layer was essentially the same as using a bad conductor such as oils of various kinds. In such a case the brush of the same polarity as the electrode also started at and extended outward directly from the electrode.

Judging from the experiments above stated, it seems now evident that a liquid of proper electric resistance was suitable in order to cause the curious discharge figure observed by Mikola.

Thus when the water from the water pipes was employed to cover the portion of the photographic plate around the electrode, a thickness of from one tenth of a millimeter to one millimeter was found to be suitable to get the curious discharge figure in question. The photograph reproduced in Fig. 2 was obtained under such a condition.

In the case of a very bad conducting liquid the electric discharge from the electrode takes place only in the form of a visible brush discharge along the surface of the liquid. On the contrary in the case of a good conducting liquid, the electric discharge from the electrode passes only through the liquid without causing any luminosity. Thus with a liquid layer of proper thickness and of intermediate conductivity, such as water, it may happen that the electric discharge takes the form of a luminous brush along a portion of the surface of the liquid, together with a non-visible discharge through the liquid itself.

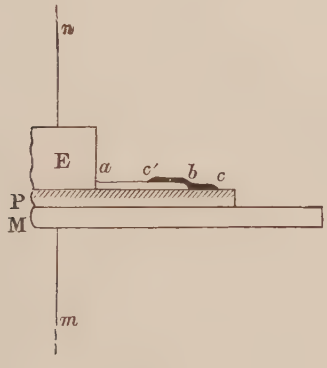


Fig. 5.

In Fig. 5, let E be the anode, M the metallic plate connected to the negative pole of an influence machine, P the photographic plate, and ab the layer of the water of proper thickness. On account of the comparatively smallness of the resistance of the liquid layer, when the potential difference between E and M is sufficiently incrsased, the positive charge on the anode E will be conducted through the water, not causing at all the brush discharge along the surface of the water. The luminous positive brush discharge will now start at the boundary of the water b and will extend outward to c along the surface of the sensitive film of the plate. In this state of the dis-

charge, the branches of the positive brush bc will behave as if it were a conductor. The positive ions will be repelled toward the terminals c of the branches of the brush, and a large number of negative ions will accumulate on the sharp starting points b of the branches. Consequently, in spite of the presence of a semiconducting liquid layer, the electric field along its surface will become so intense at the end b that the brush of the negative polarity now starts at the boundary of the liquid and extends back toward the electrode along the surface of the liquid. When the electrode E on the photographic plate is made cathode, we have simply to reverse the signs of all the charges and brushes respectively. The photographic impression of the brush which starts at the boundary b of the liquid and extends back toward the electrode seems to be due to the brush along the surface of the liquid. Thus considered, the fact that the impression of the brush under consideration becomes more diffused with a thicker liquid layer seems to be explained at once.

In conclusion, the writer wishes to express his sincere thanks to Assist. Prof. U. Yoshida for his kind guidance during the work.

Fig. 2.

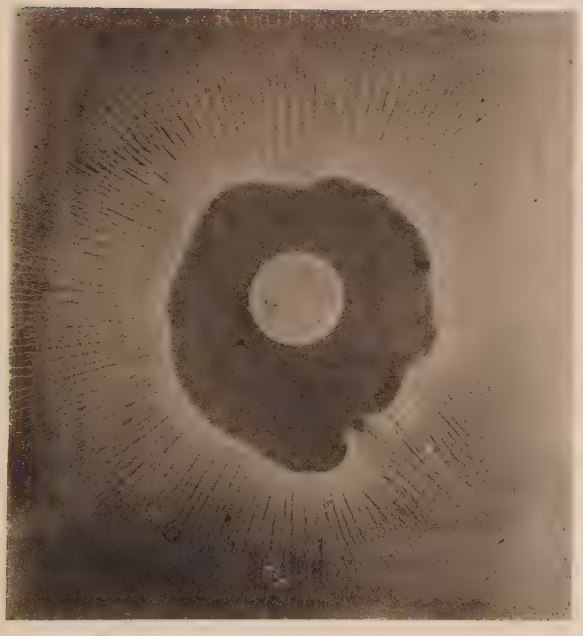


Fig. 3.

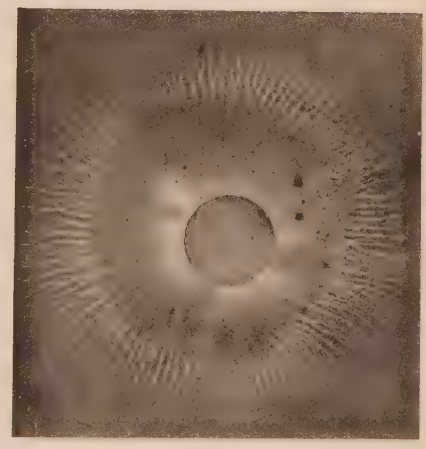


Fig. 4.

On the Solutions of Partial Differential Equations of the First Order at the Singular Points IV

By

Toshizô Matsumoto

(Received July 27, 1921)

INTRODUCTION.

In continuing my study on the solutions of partial differential equations of the first order at the singular points, I have recently met with a valuable memoir by Mr. Dulac.[1] He has completed and developed, by his own methods, the researches of several mathematicians, on the system of the ordinary differential equations of the first order:

$$\frac{dx_1}{\xi_1} = \frac{dx_2}{\xi_2} = \ldots = \frac{dx_n}{\xi_n},$$

where $\xi_1, \xi_2, \ldots, \xi_n$ commence with terms of the first order in $x_1, x_2, \ldots, x_n$. His memoir may be divided into three parts: 1°. The case where both of the Poincaré conditions[2] are satisfied; 2°. the case where the second of the conditions is valid; 3°. the case where neither condition is satisfied. For the second case he has devised a special transformation of the variables, somewhat resembling that which I used in my third paper.[3] In his memoir, sometimes such partial differential equations as

1 Bull. soc. math. Fr., t. 40 (1912).

2 My first paper; these Memoirs, vol. II, no. 5 (1917), p. 257.

3 Third paper; these Memoirs. vol. III, no. 5 (1920).

$$Xf=\xi_1\frac{\partial f}{\partial x_1}+\xi_2\frac{\partial f}{\partial x_2}+\ldots+\xi_n\frac{\partial f}{\partial x_n}=\lambda_i f+\varphi$$

are employed and sometimes an auxiliary variable t. It is not his proper intention to discuss the partial differential equation $Xf=o$. I have treated in the first part of my first paper, the case where the characteristic equation

$$\begin{vmatrix} \lambda_{11}-\lambda & \lambda_{12} & \ldots\ldots & \lambda_{1n} \\ \lambda_{21} & \lambda_{22}-\lambda & \ldots\ldots & \lambda_{2n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \lambda_{n1} & \lambda_{n2} & \ldots\ldots & \lambda_{nn}-\lambda \end{vmatrix}=o,$$

where

$$\xi_i=\lambda_{i1}x_1+\lambda_{i2}x_2+\ldots+\lambda_{in}x_n+\ldots,\quad i=1,\ 2,\ \ldots,\ n,$$

has the multiple roots $\lambda_1, \lambda_2, \ldots, \lambda_\nu$ $(\nu<n)$ and they satisfy both of Poincaré's conditions. I have solved this case directly, considering upon Xf, not by aid of changes of the variables of the system of ordinary differential equations. But in my first paper, the case where the roots $\lambda_1, \lambda_2, \ldots, \lambda_\nu$, satisfy the second of Poincaré's conditions but not the first, is not discussed. This case is solved in the second part of Dulac's memoir. Since my considerations can easily be applied, I want here to discuss the case directly by my own method. After this discussion, I shall enter the discussion of the system of partial differential equations $Xf=o$, $Yf=o$ where $(XY)f\equiv o$.

1. We transform, as usual[1], the equation $Xf=o$ by a certain linear transformation of variables such that the determinant made by the coefficients of the first orders of $\xi_1, \xi_2, \ldots, \xi_n$ becomes

$$\begin{vmatrix} L_1 & O & \ldots\ldots & O \\ O & L_2 & \ldots\ldots & O \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ O & O & \ldots\ldots & L_\nu \end{vmatrix} \qquad (1)$$

where all the elements above the diagonal are zero and the diagonal elements of L_1 are the multiple root λ_1 of order n_1, those of L_2, the multiple root λ_2 of order n_2, ..., those of L_ν, the multiple root λ_ν of order n_ν. Moreover the multiple roots $\lambda_1, \lambda_2, \ldots, \lambda_\nu$ may be arranged[2]

[1] The first paper, *loc. cit.*, p. 268.

[2] The second paper, these Memoirs, vol. IV, no. 3 (1919), p. 78.

in the increasing order of their real parts and the number of the relations such as

$$p_h\lambda_h+\ldots+p_i\lambda_i=\lambda_j, \quad 1\leq h\leq i<j, \tag{2}$$

are always finite. In the following we use the notations:

$$H\equiv n_1+n_2+\ldots+n_{h-1},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$I\equiv n_1+n_2+\ldots+n_{i-1},$$

$$J\equiv n_1+n_2+\ldots+n_{j-1},$$

and the like for other suffixes.

If λ_i can not be expressed linearly by $\lambda_1, \lambda_2, \ldots, \lambda_{i-1}$ with some positive integral coefficients (zero inclusive), then as I have proved, we may obtain n_i holomorphic solutions about $(o, o, \ldots, o)$:

$$\begin{aligned} f_{I+1}&=x_{I+1}+\ldots,\\ &\ldots\ldots\ldots\ldots\\ f_{I+n_i}&=x_{I+n_i}+\ldots, \end{aligned} \tag{3}$$

the dotted parts standing for terms of higher orders, such that

$$Xf_{I+1}=\lambda_i f_{I+1},$$

$$\ldots\ldots\ldots\ldots$$

$$Xf_{I+n_i}=\lambda_{I+n_i1}f_{I+1}+\lambda_{I+n_i2}f_{I+2}+\ldots+\lambda_i f_{I+n_i},$$

where

$$\xi_{I+1}=\lambda_i x_{I+1}+\ldots,$$

$$\ldots\ldots\ldots\ldots$$

$$\xi_{I+n_i}=\lambda_{I+n_i1}x_{I+1}+\lambda_{I+n_i2}x_{I+2}+\ldots\lambda_i x_{I+n_i}+\ldots$$

But for the equation $Xf=\lambda_j f$, the same considerations are fatal. Suppose λ_j the first coefficient which satisfies some number of relations such as (2). Differentiate the equation

$$Xf=\lambda_j f, \tag{4}$$

p_h times with respect to $x_{H+1}, \ldots, p_i$ times with respect to x_{I+1} and put $x_1=x_2=\ldots=x_n=o$, then we obtain the equation

$$(p_h\lambda_h+\ldots+p_i\lambda_i)\ (x_{H+1}^{p_h}\ldots x_{I+1}^{p_i})+\ldots\ldots$$

$$=\lambda_j(x_{H+1}^{p_h}\ldots x_{I+1}^{p_i}).$$

The dotted part stands for a linear expression of quotients of lower stages and of lower orders. Since by the relation

$$p_h\lambda_h+\ldots+p_i\lambda_i=\lambda_j,$$

we can not determine the quotient $(x_{H+1}^{p_h}\ldots x_{I+1}^{p_i})$. But if we consider the equation

$$Xf=\lambda_j f+Ax_{H+1}^{p_h}\ldots x_{I+1}^{p},$$

where A is yet undetermined, we obtain the relation

$$(p_h\lambda_h+\ldots+p_i\lambda_i)\ (x_{H+1}^{p_h}\ldots x_{I+1}^{p_i})+\ldots$$

$$=\lambda_j(x_{H+1}^{p_h}\ldots x_{I+1}^{p_i})+Ap_h!\ldots p\ !$$

The dotted part has the same meaning. Now we may take the indeterminate coefficient A such that the equation is valid.

Since $(x_{H+1}^{p_h}\ldots x_{I+1}^{p_i})$ is arbitrary, we fix

$$(x_{H+1}^{p_h}\ldots x_{I+1}^{p_i})=o.$$

2. Since the multiplicity of the root λ_h is n_h, such circumstances will also occur when we require the quotients

$$(x_{H+1}^{q}\,x_{H+2}^{r}\ldots x_{H+n_h}^{s}\ldots x_{I+1}^{p_i}),\quad q+r+\ldots+s=p_h,$$

for the coefficient of x_{H+t} $(1\leq t\leq n_h)$ in ξ_{H+t}, the coefficient of $\dfrac{\partial f}{\partial x_{H+t}}$, is λ_h. This is the same for the remaining multiple roots e.g., for λ_t. Therefore we must add to the right hand side of (4), every possible term, each multiplied by an indeterminate coefficient, such that

$$Ax_{H+1}^{q}\,x_{H+2}^{r}\ldots x_{H+n_h}^{s}\ldots x_{I+1}^{q'}\,x_{I+2}^{r'}\ldots x_{I+n_i}^{s'},$$

$$q+r+\ldots+s=p_1,\ \ldots,\ q'+r'+\ldots+s'=p_i.$$

Observing these, we consider therefore the equation

$$Xf=\lambda_j f+\sum A f_{H+1}^{q} f_{H+2}^{r} \cdots f_{H+n_h}^{s} \cdots f_{I+1}^{q'} f_{I+2}^{r'} \cdots f_{I+n_i}^{s'}. \tag{5}$$

$\sum$ denotes the sum of all possible terms such that

$$(q+r+\ldots+s)\lambda_h+\ldots+(q'+r'+\ldots+s')\lambda_i=\lambda_j. \tag{6}$$

The total number of such terms, as we have said, is finite. From quotients of the lower stages to the higher, we calculate successively by means of equation (5) and determine all the indeterminate coefficients A where we take all

$$(x_{H+1}^{q} x_{H+2}^{r} \cdots x_{H+n_h}^{s} \cdots x_{I+1}^{q'} x_{I+2}^{r'} \cdots x_{I+n_i}^{s'})=0.$$

Thus we may calculate all the quotients of f which satisfy (5). As the initial condition we take as usual.

$$(x_1)=\ldots=(x_J)=(x_{J+1})-1=(x_{J+2})=\ldots=(x_n)=0,$$

and the solution will be

$$f=f_{J+1}=x_{J+1}+\ldots,$$

the dotted part being terms of higher orders.

3. Next consider the equation

$$Xf=\lambda_j f+\lambda_{J+2J+1} f_{J+1}+\sum A' f_{H+1}^{q} f_{H+2}^{r} \cdots f_{H+n_h}^{s} \cdots f_{I+1}^{q'} f_{I+2}^{r'} \cdots f_{I+n_i}^{s'}$$

Quite in the same way, we obtain the solution

$$f=f_{J+2}=x_{J+2}+\ldots,$$

the dotted part being terms of higher orders. We continue the process up to the solution

$$f=f_{J+n_j}=x_{J+n_j}+\ldots$$

of the equation

$$Xf\equiv\lambda_j f+\lambda_{J+n_j\,J+1} f_{J+1}+\lambda_{J+n_j\,J+2} f_{J+2}+\ldots+\sum A''\ldots.$$

The proof of the convergencies of these solutions may be done quite

in the same way as stated in the first paper. For we have only to add to θ^1 the *série majorante* of such series as

$$\sum A f_{H+1}^{q} f_{H+2}^{r} \cdots f_{H+n_h}^{s} \cdots f_{I+1}^{q'} f_{I+2}^{r'} \cdots f_{I+n_i}^{s'}$$

For the other multiple roots which may be expressed linearly by the others, we may treat quite in the same way and obtain n independent holomorphic solutions.

$$f_1 = x_1 + \cdots,$$
$$\cdots\cdots\cdots,$$
$$f_n = x_n + \cdots.$$

The dotted parts stand for terms of higher orders.
Now by the transformation

$$y_i = f_i, \quad i = 1, 2, \ldots n,$$

Xf will become

$$Yf \equiv \eta_1 \frac{\partial f}{\partial y_1} + \eta_2 \frac{\partial f}{\partial y_2} + \ldots + \eta_n \frac{\partial f}{\partial y_n},$$

where

$$(7)\begin{cases} \eta_1(y) = \lambda_1 y_1, \\ \eta_2(y) = \lambda_{21} y_1 + \lambda_1 y_2, \\ \cdots\cdots\cdots\cdots \\ \eta_{n_1}(y) = \lambda_{n_1 1} y_1 + \lambda_{n_1 2} y_2 + \ldots + \lambda_1 y_{n_1}, \\ \cdots\cdots\cdots\cdots, \end{cases}$$

$$(8)\begin{cases} \eta_{J+1}(y) = \lambda_j y_{J+1} + \sum A y_{H+1}^{q} y_{H+2}^{r} \cdots y_{H+n_h}^{s} \cdots y_{I+1}^{q'} y_{I+2}^{r'} \cdots y_{I+n_i}^{s'}, \\ \eta_{J+2}(y) = \lambda_{J+2\,J+1} y_{J+1} + \lambda_j y_{J+2} + \sum A' \ldots, \\ \cdots\cdots\cdots\cdots \\ \eta_{J+n_j}(y) = \lambda_{J+n_j\,J+1} y_{J+1} + \lambda_{J+n_j\,J+2} y_{J+2} + \ldots + \lambda_j y_{J+n_j} + \sum A'' \ldots, \\ \cdots\cdots\cdots\cdots \end{cases}$$

The coefficients not written here belong to the one or the other type and the functions under the sign $\sum$ are *polynomials*. We notice that

[1] The first paper, *loc. cit.*, p. 276.

the polynomials in (8) are the same, except their coefficients A, A', ..., A''.

4. To solve the equation $Xf=o$ is thus reduced to solving the equation $Yf=o$. For this purpose consider, as it is classical, the system of the differential equations.

$$\frac{dy_1}{\eta_1}=\frac{dy_2}{\eta_2}=\ldots=\frac{dy_{n_1}}{\eta_{n_1}}=\ldots$$

$$=\frac{dy_{J+1}}{\eta_{J+1}}=\frac{dy_{J+2}}{\eta_{J+2}}=\ldots=\frac{dy_{J+n_j}}{\eta_{J+n_j}}=\ldots=dt.$$

First consider the system

$$\frac{dy_1}{\eta_1}=\frac{dy_2}{\eta_2}=\ldots=\frac{dy_{n_1}}{\eta_{n_1}}=dt.$$

From the nature of the functions $\eta_1, \eta_2, \ldots, \eta_{n_1}$, we have the solutions

$$\begin{aligned} &y_1=e^{\lambda_1 t} \\ &y_\mu=[C_\mu+poly\,(t, C_2, C_3, \ldots C_{\mu-1})]e^{\lambda_1 t},\ \mu=2, \ldots, n_1, \end{aligned} \tag{9}$$

where C_2, C_3, ..., C_μ are constants. These processes are followed for all the multiple roots $\lambda_h, \ldots, \lambda_i$ and will obtain similar solutions. Next to solve the system

$$\frac{dy_{J+1}}{\eta_{J+1}}=\frac{dy_{J+2}}{\eta_{J+2}}=\ldots=\frac{dy_{J+n_j}}{\eta_{J+n_j}}=dt,$$

at first we notice that by the relations (5) and (6), putting the values of $y_{H+1}, \ldots, y_{I+n_i}$ such as (7), we have

$$y_{H+1}^{q}\, y_{H+2}^{r} \cdots y_{H+n_h}^{s} \cdots y_{I+1}^{q'}\, y_{I+2}^{r'} \cdots y_{I+n_i}^{s'}=poly\,(t, C_{H+2}, C_{H+2}, \ldots, C_{H+n_h}, \ldots,$$

$$C_{I+1}, C_{I+2}, \ldots, C_{I+n_i})e^{\lambda_j t}$$

Therefore the solutions are

$$y_{J+\mu}=[C_{J+\mu}+poly\,(t, C_{H+1}, \ldots, C_{I+n_i};\ C_{J+1}, \ldots, C_{J+\mu-1})]e^{\lambda_j t},\ \mu=1, 2, \ldots n_j.$$

Hence the solutions of the partial differential equation $Yf=o$ are :

$$y_{M+\mu}\, y_1^{-\frac{\lambda_1}{\lambda_m}} + poly\,(\log y_1, y_{M+1}\, y_1^{-\frac{\lambda_1}{\lambda_m}}, \ldots, y_{M+\mu-1}\, y_1^{-\frac{\lambda_1}{\lambda_m}}),$$
$$M = o, \ldots, H. \ldots, I, \ldots ; \mu = 1, 2, \ldots, n_m,$$

where for $M=o$, we commence μ by 2 and for the other case, we cancel the expression *poly*, provided $\mu=1$; and

$$y_{J+\mu}\, y_1^{-\frac{\lambda_1}{\lambda_j}} + poly\,(\log y_1,\ y_{H+1}\, y_1^{-\frac{\lambda_1}{\lambda_h}}, \ldots, y_{I+n_i}\, y_1^{-\frac{\lambda_1}{\lambda_i}} ; y_{J+1} y_1^{-\frac{\lambda_1}{\lambda_j}}, \ldots, y_{J+\mu-1}\, y^{-\frac{\lambda_1}{\lambda_j}}),$$
$$J = J, \ldots ; \mu = 1, 2, \ldots, n_j.$$

When λ_l is not the first coefficient which may be expressed linearly, yet the solutions may be given by the latter forms. Thus *our proposition is fully solved* and we see that the transformation of Mr. Dulac is unnecessary. The forms of the solutions written above are allowed as the extensions of the forms of the solutions given in my first paper.[1]

5. We shall enter into the discussions of two simultaneous partial differential equations

$$Xf \equiv \xi_1 \frac{\partial f}{\partial x_1} + \xi_2 \frac{\partial f}{\partial x_2} + \ldots + \xi_n \frac{\partial f}{\partial x_n} = o,$$

$$Yf \equiv \eta_1 \frac{\partial f}{\partial x_1} + \eta_2 \frac{\partial f}{\partial x_2} + \ldots + \eta_n \frac{\partial f}{\partial x_n} = o,$$

where Xf, Yf, are permutable to each other, i.e., $(XY)f \equiv o$ and all the coefficients $\xi_1, \ldots, \eta_n$ commence by terms of the first orders In x_1, $x_2, \ldots, x_n$. Since Xf and Yf are permutable, the matrices formed by the coefficients of the first orders in the coefficients $\xi_1, \ldots, \eta_n$ are permutable and hence they may be transformed to become normal simultaneously, by a certain transformation of the variables.[2] Therefore we suppose from the beginning :

$$Xf = \xi_1 \frac{\partial f}{\partial x_1} + \xi_2 \frac{\partial f}{\partial x_2} + \ldots + \xi_n \frac{\partial f}{\partial x_n}$$

1 First paper, *loc. cit.*, p. 279.

2 First paper, *loc. cit.*, p. 280.

where using the same notations,

$$\left.\begin{aligned}
&\xi_{M+1}=\lambda_m\, x_{M+1},\\
&\xi_{M+2}=\lambda_{M+2\ M+1}\, x_{M+1}+\lambda_m\, x_{M+2},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\xi_{M+n_m}=\lambda_{M+n_m\ M+1}\, x_{M+1}+\lambda_{M+n_m\ M+2}\, x_{M+2}+\ldots+\lambda_m\, x_{M+n_m},
\end{aligned}\right\}\quad(10)$$

provided λ_m can not be expressed linearly by the others, and

$$\left.\begin{aligned}
&\xi_{J+1}=\lambda_j\, x_{J+1}+\sum C\, x_{H+1}^{p_h'}\ldots x_{H+n_h}^{p_h''}\ldots x_{I+1}^{p_i'}\ldots x_{I+n_i}^{p_i''},\\
&\xi_{J+2}=\lambda_{J+2\ J+1}\, x_{J+1}+\lambda_j\, x_{J+2}+\sum C'\ldots,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\xi_{J+n_j}=\lambda_{J+n_j\ J+1}\, x_{J+1}+\lambda_{J+n_j\ J+2}\, x_{J+2}+\ldots+\lambda_j\, x_{J+n_j}+\sum C''\ldots,
\end{aligned}\right\}\quad(11)$$

provided λ_j satisfies the relations (2). $\sum$ denotes the same polynomial, with the exception of the coefficients $C, C',\ldots, C'',\ldots$. By our transformation Yf does not change its form. So we have

$$Yf=\eta_1\frac{\partial f}{\partial x_1}+\eta_2\frac{\partial f}{\partial x_2}+\ldots+\eta_n\frac{\partial f}{\partial x_n},$$

$$\eta_l=\mu_{l1}\, x_1+\mu_{l2}\, x_2+\ldots+\mu_{ll}\, x_l+\ldots,\quad l=1, 2,\ldots, n.\qquad(12)$$

The dotted part stands for terms of higher orders. We shall determine the forms of the coefficients of Yf more closely.

6. Since Xf and Yf are permutable, corresponding to the root λ_m we have the relations

$$\left.\begin{aligned}
&X\eta_{M+1}=\lambda_m\,\eta_{M+1},\\
&X\eta_{M+2}=\lambda_m\,\eta_{M+2}+\lambda_{M+2\ M+1}\,\eta_{M+1},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&X\eta_{M+n_m}=\lambda_m\,\eta_{M+n_m}+\lambda_{M+n_m\ M+1}\,\eta_{M+1}+\ldots+\lambda_{M+n_m\ M+n_m-1}\,\eta_{M+n_m-1}.
\end{aligned}\right.$$

First we consider the equation

$$X\eta_{M+1}=\lambda_m\,\eta_{M+1}.$$

The variables belonging to λ_a, i.e., the variables which have λ_a as their coefficients of the first orders in $\xi_{A+1}, \xi_{A+2}, \ldots, \xi_{A+n_a}$ respectively, are $x_{A+1}, x_{A+2}, \ldots, x_{A+n_a}$. In the other coefficients, λ_a does not occur. Therefore between the quotients of the first orders, we have the relations

$$\lambda_a (x_{A+n_a}) = \lambda_m (x_{A+n_a}),$$

$$\lambda_a (x_{A+r}) + \ldots = (\lambda_m x_{A+r}), \qquad 1 \leqq r < n_a.$$

The dotted part stands for a linear form of quotients of lower stages. But if we notice the form of the determinant (1), the quotients contained in the form are only

$$(x_{A+1}), (x_{A+2}), \ldots, (x_{A+r-1}).$$

Since $\lambda_a \neq \lambda_m$, by the mathematical induction, we have

$$(x_{A+1}) = (x_{A+2}) = \ldots = (x_{A+n_a}) = 0.$$

Since by (12) the terms of first orders of η_{M+1} is a linear conbination of x_{M+1} and the other variables whose suffixes are less than $M+1$, we conclude that

$$\eta_{M+1} = \mu x_{M+1} + \ldots,$$

the dotted part being terms of higher orders.

Next consider the quotients of the second orders. They are all zero. For put

$$H \equiv \eta_{M+1} - \mu x_{M+1},$$

then H commences at least by terms of second orders. Hence we have the relations

$$2\lambda_\nu (x_n^2) = \lambda_m (x_n^2), \qquad 1 \leq r \leq n_a,$$

$$(\lambda_a + \lambda_b)(x_{A+r} x_{B+s}) + \ldots = \lambda_m (x_{A+r} x_{B+s}), \qquad 1 \leq s \leq n_b.$$

The dotted part stands for a linear form of quotients of lower stages. It contains no quotients of the first order. Hence all the quotients of second orders are zero. Therefore by the mathematical induction, we may conclude that all the quotients of higher orders are zero, hence we have

$$\eta_{M+1} = \mu x_{M+1}.$$

7. We consider next η_{M+t}, $(2 \leq t \leq n_m)$. Assume that

$$\eta_{M+t'} = \mu_1 x_{M+1} + \mu_2 x_{M+2} + \ldots + \mu_{t'} x_{M+t'}, \qquad 1 < t' < t. \tag{13}$$

For $t' = 1$, it is right. After the substitutions of the values of $\eta_{M+1}, \ldots, \eta_{M+t-1}$, in the equation

$$X\eta_{M+t} = \lambda_m \eta_{M+t} + \lambda_{M+t\ M+1} \eta_{M+1} + \ldots + \lambda_{M+t\ M+t-1} \eta_{M+t-1},$$

we have

$$X\eta_{M+t} = \lambda_m \eta_{M+t} + c_1 x_{M+1} + \ldots + c_{t-1} x_{M+t-1}, \tag{14}$$

where $c_1, \ldots, c_{t-1}$ are certain constants. Now we prove that η_{M+t} has the same form. First, it does not contain x_{A+r}, $(A < M)$. Since as before, after differentiations, we have

$$\lambda_a (x_{A+n_a}) = \lambda_m (x_{A+n_a}),$$

$$\lambda_a (x_{A+r}) + \ldots = \lambda_m (x_{A+r}), \qquad 1 \leq r < n_a,$$

the dotted part being quotients of lower stages. Hence we are right. Secondly, since η_{M+t} has none of $x_{M+t+1}, \ldots, x_n$, in the first orders, it has except the terms of higher orders the same form with what is assumed. Thirdly, put

$$H \equiv \eta_{M+t} - \textit{terms of first orders of } \eta_{M+t}.$$

Then we have, by (14),

$$XH = \lambda_m H + c_1' x_{M+1} + \ldots + c_t' x_{M+t},$$

where $c_1', \ldots, c'$ are certain constants. Now all the quotients of the first order of H are zero. Therefore noticing that the quotients of the second order satisfy the relations

$$2\lambda_r (x_n^2) = \lambda_m (x_n^2), \qquad 1 \leq r \leq n_a,$$

$$(\lambda_a + \lambda_b)(x_{A+r}\, x_{B+s}) + \ldots = \lambda_m (x_{A+r}\, x_{B+s}), \qquad 1 \leq s \leq n_b,$$

the dotted part being quotients of lower orders, we way conclude that all the quotients of the second orders are zero. Continuing this consideration, by mathematical induction, we may prove that all the quotients of the higher orders are zero. Hence

$$H \equiv o,$$

i.e., η_{M+t} has the same form as (13). Therefore, all the coefficients $\eta_{M+1}, \eta_{M+2}, \ldots, \eta_{M+n_m}$ have the same forms as $\xi_{M+1}, \xi_{M+2}, \ldots, \xi_{M+n_m}$ respectively.

8. Proceding further, consider the case of the multiple root λ_j which satisfies the relation (6) and the coefficients corresponding to it are given by (11), where

$$\begin{aligned} &p_h' + \ldots + p_h'' = p_h, && 0 \leqq p_h' \leqq p_h, \\ & && \ldots\ldots\ldots\ldots \\ & && 0 \leqq p_h'' \leqq p_h, \\ &\ldots\ldots\ldots\ldots\ldots && \ldots\ldots\ldots\ldots \\ &p_i' + \ldots + p_i'' = p_i, && 0 \leqq p_i' \leqq p_i, \\ & && \ldots\ldots\ldots\ldots \\ & && 0 \leqq p_i'' \leqq p_i. \end{aligned}$$

We assume as before $\lambda_h, \ldots, \lambda_i$ can not be represented by the others. $\eta_{J+1}, \eta_{J+2}, \ldots, \eta_{J+n_j}$ must satisfy the relations

$$\left.\begin{aligned} &X\eta_{J+1} = \lambda_j \eta_{J+1} + \sum', \\ &X\eta_{J+2} = \lambda_j \eta_{J+2} + \lambda_{J+2\,J+1}\eta_{J+1} + \sum', \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &X\eta_{J+n_j} = \lambda_j \eta_{J+n_j} + \lambda_{J+n_j\,J+1}\eta_{J+1} + \ldots + \lambda_{J+n_j\,J+n_j-1}\eta_{J+n_j-1} + \sum'. \end{aligned}\right\} \quad (15)$$

The $\sum'$ in the first equation is

$$\begin{aligned} \sum' &\equiv \sum CY(x_{H+1}^{p_h'} \ldots x_{H+n_h}^{p_h''} \ldots x_{I+1}^{p_i'} \ldots x_{I+n_i}^{p_i''}) \\ &= \sum C \left\{ p_h' \eta_{H+1} x_{H+1}^{p_h'-1} \ldots x_{H+n_h}^{p_h''} \ldots x_{I+1}^{p_i'} \ldots x_{I+n_i}^{p_i''} \right. \\ &\qquad \left. + \ldots + p_i'' \eta_{I+n_i} x_{H+1}^{p_h'} \ldots x_{H+n_h}^{p_h''} \ldots x_{I+1}^{p_i'} \ldots x_{I+n_i}^{p_i''-1} \right\}, \end{aligned} \quad (16)$$

and the like expressions for the other $\sum'$. In the following, we shall use for products such as $x_r^{p_r} \ldots x_s^{p_s} \ldots x_t^{p_t}$ the same definiton of *stages*[1].

[1] The first paper, *loc. cit.*, p. 270.

Then from the forms of $\eta_{H+1}, \ldots, \eta_{I+n_i}$, all the products under the sign $\sum'$ are respectively of same or higher stages than those under the sign $\sum$ of (11). All the products are constructed by the variables $x_{H+1}, \ldots, x_{I+n_i}$. For example, the first equation of (15) is the consequence of

$$Xf_{J+1} \equiv Y\xi_{J+1}.$$

Since by the proof of the preceding sections, we have

$$\eta_{H+1} = \mu x_{H+1},$$

$$\ldots\ldots\ldots\ldots,$$

$$\ldots\ldots\ldots\ldots,$$

$$\eta_{I+n_i} = \mu_1 x_{I+1} + \ldots + \mu_{n_i} x_{I+n_i},$$

the calculation (16) of $\sum'$ teaches us that our statement is right.

9. After these considerations, we discuss the forms of $\eta_{J+1}, \ldots, \eta_{J+n_j}$ which satisfy the equations (15). Since the polynomials of orders higher than the first in the coefficients $\xi's$ of Xf can have effect only upon the terms of higher orders of the coefficients $\eta's$ of Yf, the terms of the first order in $\eta_{J+1}, \eta_{J+2}, \ldots, \eta_{J+n_j}$ have the same forms as (13), namely

$$\eta_{J+t} = \rho_1 x_{J+1} + \rho_2 x_{J+2} + \ldots + \rho_t x_{J+t} + \ldots, \qquad 1 \leq t \leq n_j,$$

$\rho_1\ \rho_2 \ldots, \rho_t$ are certain constants and the dotted part stands for terms of higher orders. Specially we have

$$\eta_{J+1} = \rho x_{J+1} + \ldots,$$

all the quotients of the first order, except (x_{J+1}), being zero. First consider the equation

$$Xf_{J+1} = \lambda_j \eta_{J+1} + \sum'. \tag{17}$$

1°. When at least one of A and B is different from any of $H, \ldots, I$, then we have

$$(\lambda_a + \lambda_b)(x_{A+n_a} x_{B+n_b}) = \lambda_m (x_{A+n_a} x_{B+n_b}).$$

For no coefficients except ξ_{A+n_a}, ξ_{B+n_b} have the variables x_{A+n_a}, x_{B+n_b}

in their first orders and moreover the coefficient ξ_{J+1} of $\frac{\partial f}{\partial x_{J+1}}$ lacks at least one of these variables. Hence the quotient $(x_{A+n_a}\, x_{B+n_b})$ is zero. 2°. Moreover we have

$$(\lambda_a+\lambda_b)\,(x_{A+r}\, x_{B+s})+\ldots=\lambda_j\,(x_{A+r}\, x_{B+s}),$$

the dotted part being quotients of lower stages. For in the expressions of the first order, x_{A+r} appears only in $\xi_{A+r}, \ldots, \xi_{A+n_a}$ and x_{B+s} only in $\xi_{B+s}, \ldots, \xi_{B+n_b}$, and the quotients in the dotted part are of the forms

$$(x_{A+r'}\, x_{B+s'}), \qquad r<r'\leqq n_a,$$
$$s<s'\leqq n_b.$$

Therefore by mathematical induction all the quotients of the second orders $(x_{A+r}\, x_{B+s})$ are zero where at least one of A and B is different from any one of $H, \ldots, I$. 3°. We may go further and conclude that all the quotients

$$(\overset{u}{x}_{A+n_a}\overset{v}{x}_{B+n_b}\ldots\overset{w}{x}_{C+n_c})$$

of order $u+v+\ldots+w$ are zero, provided at least one of $A, B, \ldots, C$ is different from any one of $H, \ldots, I$. Moreover we have

$$(u\lambda_a+v\lambda_b+\ldots+w\lambda_c)\,(\overset{u}{x}_{A+r}\overset{v}{x}_{B+s}\ldots\overset{w}{x}_{C+t})+\ldots$$
$$=\lambda_j\,(\overset{u}{x}_{A+r}\overset{v}{x}_{B+s}\ldots\overset{w}{x}_{C+t}),$$

the dotted part being quotients of lower stages. The considerations are quite the same, ... Thus applying the mathematical induction again and again, we conclude that all the quotients

$$(\overset{u}{x}_{A+r}\overset{v}{x}_{B+s}\ldots\overset{w}{x}_{C+t})=0,$$

where at least one of $A, B, \ldots, C$ is different from $H, \ldots, I$.

10. Let us put

$$H\equiv\eta_{J+1}-\rho x_{J+1}-\sum'',$$

where $\sum''$ is the polynomial in η_{J+1} constituting the terms of orders not higher than any of those of $\sum'$. Substituting this in (17), we have

$$XH \equiv \lambda_j H + \lambda_j \sum'' + \sum' - \rho \sum - X \sum''.$$

Since $\sum''$ contains only the variables $x_{H+1}, \ldots, x_{H+n_\lambda}, \ldots, x_{I+1}, \ldots, x_{I+n}$ and by the properties of the coefficients of Xf, $X\sum''$ is of the same order as $\sum''$. Since the order of the terms of the lowest order of H and hence of XH is higher than any of $\sum$, $\sum'$, $\sum''$, we must have

$$XH \equiv \lambda_j H,$$

whence we may easily conclude that $H \equiv o$ and therefore we have

$$\eta_{J+1} = \rho x_{J+1} + \sum'',$$

$\sum''$ being of the same form as $\sum$ and $\sum'$.

11. Let us proceed to the discussion of η_{J+2}. Consider the second equation of (15)

$$X\eta_{J+2} = \lambda_j \eta_{J+2} + \lambda_{J+2\,J+1} \eta_{J+1} + \sum'.$$

Putting the value of η_{J+1} we have

$$X\eta_{J+2} = \lambda_j \eta_{J+2} + \rho \lambda_{J+2\,J+1} x_{J+1} + \sum''',$$

where $\sum'''$ is a similar polynomial as $\sum'$, $\sum''$. As we have said

$$\eta_{J+2} = \rho_1 x_{J+1} + \rho_2 x_{J+2} + \ldots,$$

the dotted part being terms of higher orders. To treat in the same way, put

$$\eta'_{J+2} \equiv \eta_{J+2} - \rho_1 x_{J+1}, - \rho_2 x_{J+2}$$

then we have

$$X\eta'_{J+2} = \lambda_j \eta'_{J+2} + \sum_1'''',$$

where by the forms of ξ_{J+1}, ξ_{J+2}, $\sum_1'''$ has the same meaning as $\sum'''$. Now we have only to repeat the processes carried out in the preceding sections and the same result must follow. Continuing these considerations, we conclude that

$$\eta_{J+1}=\rho\, x_{J+1}+\sum\nolimits_1,$$

$$\eta_{J+2}=\rho_1\, x_{J+1}+\rho_2\, x_{J+2}+\sum\nolimits_2,$$

.....

$$\eta_{J+n_j}=\rho'_1\, x_{J+1}+\rho'_2\, x_{J+2}+\ldots+\rho_{n_j}\, x_{J+n_j}+\sum\nolimits_{n_j},$$

where ρ, ρ_1, ..., ρ_{n_j} are certain constants and all the $\sum$ indicate certain polynomials of the variables x_{H+1}, ..., x_{H+n_h}, ..., x_{I+1}, ..., x_{I+n_i} and of the same order as $\sum$.

12. Proceeding further, consider the case for a multiple root λ_l which is expressible by the other, some number of which are already expressible by the other. Suppose for simplicity we have such relations as

$$p_g\,\lambda_g+\ldots+p_j\,\lambda_j+\ldots+p_k\,\lambda_k=\lambda_l,$$

where λ_g, ..., λ_k except λ_j are not expressible by the other. For λ_j we have as before the relations such as

$$p_h\,\lambda_h+\ldots+p_i\,\lambda_i=\lambda_j.$$

By these relations we see easily that in the coefficient ξ_{L+1} there are such products as

$$x_{G+1}^{p'_g}\ldots x_{G+n_g}^{p''_g}\ldots\{x_{H+1}^{p'_h}\ldots x_{H+n_h}^{p''_h}\ldots x_{I+1}^{p'_i}\ldots x_{I+n_i}^{p''_i}\}^{p_j}\ldots x_{K+1}^{p'_k}\ldots x_{K+n_k}^{p''_k}, \tag{18}$$

$$x_{G+1}^{p'_g}\ldots x_{G+n_g}^{p''_g}\ldots\{x_{H+1}^{p'_h}\ldots x_{H+n_h}^{p''_h}\ldots x_{I+1}^{p'_i}\ldots x_{I+n_i}^{p''_i}\}^{p_{j-m}}\, x_{J+1}^{m}\ldots x_{K+1}^{p'_k}\ldots x_{K+n_k}^{p''_k}, \tag{19}$$

$$p'+\ldots+p''_t=p_t,$$

where m is a positive integer. If $m=o$ (19) becomes (18). Therefore if we calculate the relation

$$Xf_{L+1}=Y\xi_{L+1},$$

we have

$$Xf_{L+1}=\lambda_l\,\eta_{L+1}+S_1+S_2,$$

where S_1 is obtained by putting such products as (18) in Yf and S_2, by putting such products as (19). If we put the product (18) in Yf, the result is a polynomial of the same order as (18). For all the

coefficients which are introduced into the polynomial are of the first order. But if we put the product (19) in Yf, then, since the coefficients $\eta_{J+1}, \ldots, \eta_{J+n_j}$, as we have proved, contain such products as

$$\frac{p'_h, \quad p''_h, \quad p'_i, \quad p''_i,}{x_{H+1} \ldots x_{H+n_h} \ldots x_{I+1} \ldots x_{I+n_i},}$$

the resulting polynomial is at most of order

$$p_g + \ldots + (p_h + \ldots + p_i)(p_j - m) + (m-1) + (p_h + \ldots + p_i) + \ldots + p_k$$

$$= p_g + \ldots + (p_h + \ldots + p_i)(p_j - m + 1) + (m-1) + \ldots + p_k.$$

Thus the polynomial $S_1 + S_2$ is of the same form as the polynomial in the coefficient ξ_{L+1}; and the like for the others.

Moreover, to calculate the coefficients η_{L+t}, we may, as we have done many times, conclude that all of their quotients which imply some variables other than $x_{L+1}, \ldots, x_{L+n_l}$ are zero. Thus we may conclude that all the coefficients $\eta_{L+1}, \ldots, \eta_{L+n_l}$, have the same forms as the cocfficients $\xi_{L+1}, \ldots, \xi_{L+n_l}$, respectively.

13. We remark that for any suffix, all the coefficients corresponding to $\xi_{M+1}, \xi_{M+2}, \ldots, \xi_{M+n_m}$, with respect to their terms of the first order, have the forms:

$$\eta_{M+1} = \mu_m x_{M+1} + \ldots,$$

$$\eta_{M+2} = \mu_{M+2\ M+1} x_{M+1} + \mu_m x_{M+2} + \ldots,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\eta_{M+n_m} = \mu_{M+n_m\ M+1} x_{M+1} + \ldots + \mu_m x_{M+n_m} + \ldots$$

The multiple roots $\mu_1, \ldots, \mu_\nu$ are different with the exception of some special cases. Concise knowledge about these can not be obtained from the idea of the permutability of two matrices. We may conclude these results above obtained as follows:—*When the infinitesimal transformations Xf and Yf are permutable with each other, where their coefficients commence by terms of the first order and the characteristic equation made by the coefficients of the first order in those of Xf has multiple roots which satisfy the second of Poincaré's conditions, then the other iufinitesimal transformation Yf has exactly the same form as the infinitesimal transformation Xf.* The special case of this was proved

in my first paper. We shall not here discuss the forms of the common solutions of the simultaneous partial differential equations, for, after the reductions have been made the next task is easy. In the following, a simple example, to explain §12 is given.

12. Consider the transformations

$$Xf \equiv x\frac{\partial f}{\partial x} + (2y + x^2)\frac{\partial f}{\partial y} + (3z + 2xy + x^3)\frac{\partial f}{\partial z},$$

$$Yf \equiv \xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y} + \zeta\frac{\partial f}{\partial z},$$

where $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$, so that $2\lambda_1 = \lambda_2$, $3\lambda_1 = \lambda_3$, $\lambda_1 + \lambda_2 = \lambda_3$. For the permutability we must have

1°. $X\xi = \xi$,

2°. $X\eta = 2\eta + 2x\xi$,

3°. $X\zeta = 3\zeta + 2(\xi y + \eta x) + 3x^2\xi$

From 1°. we obtain

$$\xi = \lambda x,$$

where λ is arbitrary. Substituting in 2°. we have from the equation

$$X\eta = 2\eta + 2\lambda x^2,$$

that

$$\eta = 2\lambda y + \nu x^2,$$

where ν is arbitrary. Substituting in the third equation, we have

$$X\zeta = 3\zeta + 6\lambda xy + (2\nu + 3\lambda)x^3,$$

from which we have

$$\zeta = 3\lambda z + 2\lambda xy + \omega x^3,$$

where ω is arbitrary.

Therefore the required expression for Yf is

$$Yf = \lambda x\frac{\partial f}{\partial x} + (2\lambda y + \nu x^2)\frac{\partial f}{\partial y} + (3\lambda z + 2\nu xy + \omega x^3)\frac{\partial f}{\partial z},$$

λ, ν, ω being arbitrary.

Thus we notice the coefficients have the same forms as those of Xf.

The solutions of the equation $Xf=o$ are

$$f_2 \equiv yx^{-2} - log\ x,\quad f_3 \equiv zx^{-3} - (2yx^{-2}+1)\ log\ x + (log\ x)^2.$$

The common solution of the simultaneous equations is the arbitrary function of $f_3 - f_2^2 - \frac{\omega-\lambda}{\nu-\lambda} f_2$, provided $\nu \neq \lambda$.

Two permutable linear homogeneous transformations have at least a common pole. What is the case for Lie's group? This is the motive of my study. About ordinary points, the study is clear. So we have to study the singular points. By these successive papers, I think, the properties became a little clearer.

Catalytic Reduction of Dextro-Camphor. I.

By

Shigeru Komatsu and Bunkichi Masumoto.

(Received September 18, 1921).

It is a well known fact that ketones by a catalytic reduction by the method of Professor Paul Sabatier, the >CO group in the molecule is reduced to the >CHOH group. In this manner aliphatic ketone can be transformed into secondary alcohol; aromatic ketone, on the other hand, tends to yield hydrocarbon.[1] Such different results in the reduction of aliphatic and aromatic compounds may be explained by assuming that aromatic alcohol formed during the process of reduction of the ketones is less stable toward heat than aliphatic alcohol and consequently the dehydration of the former taking place with the formation of hydrocarbon and water. Extending this assumption, we can naturally imagine a simultaneous formation of alcohol and hydrocarbon from a ketone of the polymethylene series by catalytic reduction below 180°. And such expectation was realized by the experiments of A. Skita and H. Ritter on cyclohexanone[2]; and N. Zelinsky on methylcyclopentanone.[3] It is therefore, anticipated that camphor by the catalytic reduction can be transformed into camphane and borneol.

W. Ipatiew has stated that when dextro-camphor was reduced with hydrogen in presence of nickel oxide under 100 atmospheric pressure and 320–350°, dextro-borneol was obtained[4] and also that

1 P. Sabatier: La Catalyse en Chimie Organique, 1920, p. 145.

2 Ber. d. Chem. Gessel., **44**, 668 (1911).

3 Ibid., **44**, 2781 (1911).

4 A. Skita: Über Katalytische Reduktionen Organischer Verbindungen, 1912, S. 68.

isocamphane was formed by subjecting camphor, borneol, isobornel and camphene respectively to the action of hydrogen in presence of nickel oxide and alumina at 200–240° and high pressure.[1]

Referring to the literature[2] we have learned that the results of Ipatiew's work are on the whole very similar to those obtained by Professor Sabatier's process.

The present investigation was undertaken by the authors to ascertain whether borneol, camphane or isocomphane could be obtained from camphor by catalytic hydrogenation.

Dextro-camphor (Pharm. Jap.), purified by sublimation, melted at 176° and showing $[\alpha]_D = +39°10'$ in alcohol solution, was passed with hydrogen generated with zinc and sulphuric acid and purified by washing with concentrated sulphuric acid and concentrated sodium hydroxide solution successively and by passing on heated copper and then dried with caustic potash, on reduced nickel heated 175–185° which was prepared according to the direction of Professor Sabatier.[3] Thus 73 gr. of the product were obtained from 105 gr. dextro-camphor. It is a white crystalline substance, and is easily sublimable at ordinary temperature. To purify, 40 gr. of the product were subjected to fractional distillation for 5 times under atmospheric pressure and the following results were obtained:

Ist. distillation under 752 m.m. pressure.

Fraction:	165–168°,	168–177°,	above 177°
Yield:	17 gr.	20 gr.	1.5 gr.

IVth. distillation under 757 m.m. pressure.

Fraction:	164–166°,	166–168°,	168–170°,	170–175°,	above 175°,
Yield:	18.5 gr.	13.7 gr.	5 gr.	5 gr.	3 gr.

Vth. distillation under 75 7 m.m. pressure.

Fraction:	164–165°,	165–168°,	168–170°,	170–175°,	above 175°,
Yield:	27.5 gr.	9 gr.	1.5 gr.	2 gr.	—gr.

The melting point and optical rotatory power of each fraction were determined, and the results are shown in the following table with the results of their analysis:

---

1 Ber. d. Chem. Gesell., **45**, 3208 (1912).

2 P. Sabatier: La Catalyse en Chimie Organique, 1920; A. Skita: Über Katalytische Reduktionen Organischer Verbindungen, 1912; G. Woker: Die Katalyse, 1915; G. G. Henderson: Catalysis in Industrial Chemistry, 1919.

3 Loc-cit.

Fraction:	164–165°,	165–168°,	168–170°,	170–175°,	175°–
M. p.	65°	63°	76°	82°	170°

Fraction, 164–165°:

0.206 gr. substance, dissolved in alcohol, made up to 10 c.c., gave $\alpha = -9'$ in 10 c.m. tube; whence $[\alpha]_D = -7°.17'$.

I. 0.1532 gr. substance gave 0.4802 gr. CO_2 and 0.1795 gr. H_2O.
II. 0.1585 „ 0.4978 „ 0.1848 „
III. 0.1548 „ 0.4840 „ 0.1816 „
IV. 0.1551 „ 0.4873 „ 0.1827 „
V. 0.1524 „ 0.4774 „ 0.1790 „

	Calc. for	Found.				
	$C_{10}H_{18}$.	I.	II.	III.	IV.	V
Carbon	86.87	85.48	85.65	85.27	85.68	85.43
Hydrogen	13.13	13.11	13.05	13.13	13.18	13.14

Fraction, 165–168°:

0.2106 gr. substance, dissolved in alcohol, made up to 10 c.c., gave $\alpha = -2'$ in 10 c.m. tube; whence $[\alpha]_D = -1°.35'$.

I. 0.1590 gr. substance gave 0.4994 gr. CO_2 and 0.1800 gr. H_2O.
II. 0.1534 „ 0.4814 „ 0.1787 „
III. 0.1556 „ 0.4907 „ 0.1920 „
IV. 0.1556 „ 0.4886 „ 0.1898 „

	Calc. for	Found.			
	$C_{10}H_{18}$.	I.	II.	III.	IV.
Carbon	86.87	85.65	85.58	86.00	85.63
Hydrogen	13.13	12.67	13.04	13.81	13.65

Fraction, 168–170°:

0 2 gr. substance, dissolved in alcohol, made up to 10. c.c., gave $\alpha = 0$ in 10 c.m. tube.

I. 0.1274 gr. substance gave 0.3944 gr. CO_2 and 0.1471 gr. H_2O.
II. 0.1278 „ 0.4009 „ 0.1494 „
III. 0.1408 „ 0.4400 „ 0.1615 „

	Calc. for		Found.		
	$C_{10}H_{18}$.	$C_{10}H_{18}O$.	I.	II.	III.
Carbon	86.87	77.83	84.42	85.55	85.22
Hydrogen	13.13	11.77	12.92	13.08	12.84

Fraction, 170–175°:

0.1559 gr. substance, dissolved in alcohol, made up to 10 c.c., gave $\alpha = +8'$ in 10 c.m. tube; whence $[\alpha]_D = +8°.33'$.

I. 0.1596 gr. substance gave 0.4964 gr. CO_2 and 0.1819 gr. H_2O.
II. 0.1868 ,, 0.5773 ,, 0.2106 ,,

	Calc. for		Found.	
	$C_{10}H_{18}$.	$C_{10}H_{18}O$.	I.	II.
Carbon	86.87	77.83	84.82	84.28
Hydrogen	13.13	11.77	12.85	12.57

Fraction, above 175°:

0.2 gr. substance, dissolved in alcohol, made up to 10 c.c., gave $\alpha = +26'$, in 10 c.m. tube; whence $[\alpha]_D = +21°.40'$.

I. 0.1571 gr. substance gave 0.4528 gr. CO_2 and 0.1585 gr. H_2O.
II. 0.1291 ,, 0.3729 ,, 0.1288 ,,

	Calc. for		Found.	
	$C_{10}H_{18}$.	$C_{10}H_{18}O$.	I.	II.
Carbon	86.87	77.83	78.60	78.98
Hydrogen	13.13	11.77	11.29	11.17

As seen in the foregoing table, the first fraction consists of a hydrocarbon $C_{10}H_{18}$ requires C. 86,87%; H. 13.13% and the main portion of the last fraction is a mixture of *d*-borneol and unchanged *d*-camphor which requires C. 77.83%; H. 11.77% and C. 78 85%; H. 10.60% respectively, and other fractions contain both the hydrocarbon and the alcohol in different proportions.

The hydrocarbon of the first fraction, a white crystalline substance, has a pleasant odour different from that of camphor or borneol, and is easily soluble in orgainic solvents but insoluble in water. From its chemical and physical properties so far as we studied, it seems to be identical with isocamphane obtained by P. Lipp[1] from camphene by catalytic hydrogenation with nickel at 180°. For confirmation, the substance was subjected to oxidation with nitric acid following Lipp's direction and carboxylic acid of apocamphoric acid was obtained which separated from the hydrocarbon unchanged and an oily product which formed simultaneously during the oxidation of the substance, was analysed of its silver salt:

0.1769 gr. substance gave 0.1203gr.CO_2, 0.0361gr.H_2O and 0.1069 gr. Ag.

	Found	Calc. for $C_{10}H_{11}O_6Ag_3$
Carbon	18.55	21.78

[1] Ann. d. Chem., **382**, 279 (1911).

Hydrogen	2.28	2.02
Silver	59.86	58.78

30 gr. of laevo-borneol of melting point 207° and of $[\alpha]_D = -39°31'$ in alcohol solution were passed with hydrogen on the reduced nickel heated at 180°, and 24 gr. of white crystalline substance were obtained.

17 gr. of the product were also subjected to fractional distillation on metallic sodium under the pressure of 758 m.m..

Ist. distillation:

Fraction:	160–170°,	175–206°
Yield:	11.5 gr.	5 gr.

IInd. distillation:

Fraction:	164–165°,	165–168°,	168–170°,	above 170°
Yield:	6 gr.	2.5 gr.	0.5 gr.	2 gr.
M. p.	63–63.5°	61°	69°	110°

Fraction, 164–165°:

0.2 gr. substance, dissolved in alcohol, made up to 10 c.c., gave $\alpha = -9'$ in 10 c.m. tube; whence $[\alpha]_D = -7°.30'$.

I. 0.1895 gr. substance gave 0.5973 gr. CO_2 and 0.2213 gr. H_2O.

II. 0.1382 „ 0.4335 „ 0.1597 „

	Calc. for $C_{10}H_{18}$.	Found. I.	Found. II.
Carbon	86.87	85.96	85.54
Hydrogen	13.13	13.07	12.93

Fraction, 165–168°:

0.1621 gr. substance, dissolved in alcohol, made up to 10 c.c., gave $\alpha = 0$ in 10 c.m. tube.

I. 0.1415 gr. substance gave 0.4439 gr. CO_2 and 0.1647 gr. H_2O.

II. 0.1435 „ 0.4491 „ 0.1629 „

	Calc. for $C_{10}H_{18}$.	Found. I.	Found. II.
Carbon	86.87	85.55	85.35
Hydrogen	13.13	13.02	12.70

Fraction, 168–170°:

0.1933 gr. substance, dissolved in alcohol, made up to 10 c.c., gave $\alpha = 0$ in 10 c.m. tube.

0.1359 gr. substance gave 0.4221 gr. CO_2 and 0.1542 gr. H_2O.

	Calc. for $C_{10}H_{18}$.	Found.
Carbon	86.87	84.71
Hydrogen	13.13	12.70

Fraction, above 170° :

0.2043 gr. substance, dissolved in alcohol, made up to 10 c.c., gave $\alpha = -12'$ in 10 c.m. tube ; whence $[\alpha]_D = -9°.47'$.

I. 0.1701 gr. substance gave 0.5198 gr. CO_2 and 0.1872 gr. H_2O.

II. 0.1503 „ 0.4582 „ 0.1647 „

	Calc. for		Found.	
	$C_{10}H_{18}$.	$C_{10}H_{18}O$.	I.	II.
Carbon	86.87	77.83	83.34	83.14
Hydrogen	13.13	11.77	12.32	12.26

As seen in the above table, laevo-isocamphane is the main reaction product from laevo-borneol by passing it with hydrogen on the heated nickel.

It is quite noteworthy that laevo-isocamphane is also obtained from dextro-borneol by the same treatment as in the cases of laevo-borneol and dextro-camphor.

25 gr. dextro-borneol, prepared from dextro-camphor by reduction with metallic sodium in its alcohol solution, melted at 199° and showing $[\alpha]_D = +28°23'$ in alcohol solution, yielded 19 gr. of white crystalline substance by passing the dextro-borneol with hydrogen on reduced nickel heated at 180°. The product was purified by fractional distillation on metallic sodium and 3.2 gr. laevo-isocamphane of b. p.: 164–165° under 753 m.m., m.p.: 65–66° were obtained.

Ist. distillation under 753 m.m. pressure.

Fraction :	164–174°,	above 174°
Yield :	14 gr.	3 gr.

IInd. distillation :

Fraction :	164–165°,	165–166°,	166–168°,	168–174°,	174°–
Yield :	3.2 gr.	4.4 gr.	3.8 gr.	1.5 gr.	1 gr.
M. p. :	65–66°	65°	71°	86°	—

Fraction, 164–165° :

0.1984 gr. substance, dissolved in alcohol, made up to 10 c.c., gave $\alpha = -5'$ in 10 c.m. tube; whence $[\alpha]_D = -4°.12'$.

0 1602 gr. substance gave 0.5040 gr. CO_2 and 0.1893 gr. H_2O.

	Calc. for $C_{10}H_{18}$.	Found.
Carbon	86.87	85.80
Hydrogen	13.13	13.22

Fraction, 165–166° :

0.2297 gr. substance, dissolved in alcohol, made up to 10 c.c., gave $\alpha = -4'$ in 10 c.m. tube; whence $[\alpha]_D = -2°.54'$.

0.1803 gr. substance gave 0.5667 gr. CO_2 and 0.212 gr. H_2O.

	Calc. for $C_{10}H_{18}$.	Found.
Carbon	86.87	85.71
Hydrogen	13.13	13.16

Fraction, 166–168° :

0.2057 gr. substance, dissolved in alcohol, made up to 10 c.c., gave $\alpha = +3'$ in 10 c.m. tube ; whence $[\alpha]_D = +2°.26'$.

0.1449 gr. substance gave 0.4560 gr. CO_2 and 0.1724 gr. H_2O.

	Calc. for $C_{10}H_{18}$.	Found.
Carbon	86.87	85.82
Hydrogen	13.13	13.31

Fraction, 168–174° :

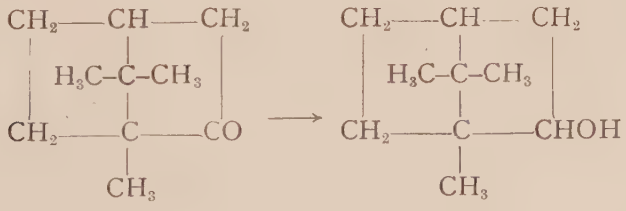
0.2095 gr. substance, dissolved in alcohol, made up to 10 c.c., gave $\alpha = +7'$ in 10 c.m. tube ; whence $[\alpha]_D = +6°.34'$.

0.1693 gr. substance gave 0.5273 gr. CO_2 and 0.1994 gr. H_2O.

	Calc. for $C_{10}H_{18}$.	Found.
Carbon	86.83	84.94
Hydrogen	13.13	13.18

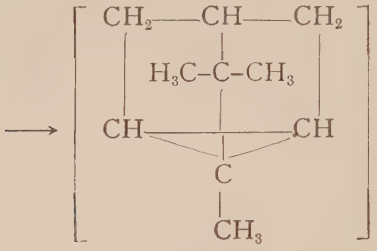
From these experimental results it would appear that the main portion of dextro-camphor by the catalytic hydrogenation in presence of reduced nickel heated at 180° can transform into laevo-isocamphane but not into camphane.

The formation of laevo-isocamphane from dextro-camphor, dextro- and laevo-borneol respectively and also that of isocampane from camphene observed by Lipp[1] naturally lead us to propose the mechanism of the transformation of camphor into isocamphane by the catalytic hydrogenation, as will be indicated by the following scheme :

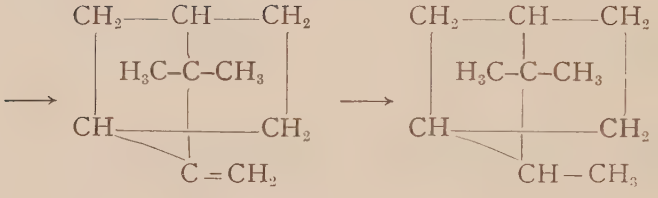


camphor borneol

[1] Lec. cit.



intermediate product



camphene isocamphane

We do not dare, however, to introduce any assumption in which an explanation is made for such an extraordinary case in stereochemistry and further work to prove laevo-isocamphane is derived from dextro-and laevo-borneol, will be attempted.

Congruence of Circles in Non-Euclidean Space.

By

Hidetoshi Kashiwagi.

(Received September 10, 1921.)

CHAPTER I.

CONGRUENCE OF CIRCLES.

§ 1. Coordinates of a Point on a Circle.

Any circle in non-euclidean space may be given by the simultaneous equations

$$(ax)^2 = \cos\frac{R}{k}(xx)(aa), \tag{1}$$

$$(bx) = 0,$$

where

$$(aa) = k^2,\ (bb) = k^2,\ (ab) = 0,$$

and (a) is the coordinates of the center of the main sphere,[1] (b) that of the plane of the circle, R the radius of the main sphere and $\frac{1}{k^2}$ the measure of curvature of the space. The center of the circle coincides with the center of the main sphere.

(1), See T. Nishiuchi, 'Oriented Circles in Non-Euclidean Space,' these Memoirs, Vol. iv, no. 6, p. 273.....

When a_0, a_1, a_2, a_3; b_0, b_1, b_2, b_3; R are functions of two parameters u and v, we have a congruence of circles, by allowing unlimited variations to the parameters.

Let (a') be the coordinates of a point C' on the plane of the circle which is orthogonal to the center C of the circle and $(a'a')=k_i^2$, then

$$(aa')=0, \quad (a'b)=0,$$

and the coordinates of any point on the circle may be given by the equations

$$\begin{aligned} x_0&=a_0\cos\frac{R}{k}+a_0'\sin\frac{R}{k},\\ x_1&=a_1\cos\frac{R}{k}+a_1'\sin\frac{R}{k},\\ x_2&=a_2\cos\frac{R}{k}+a_2'\sin\frac{R}{k},\\ x_3&=a_3\cos\frac{R}{k}+a_3'\sin\frac{R}{k}. \end{aligned} \tag{2}$$

Next, consider an absolute polar triangle in the plane of a circle of which one of the vertices is the center C of the circle, and the remaining two vertices A and B, then the point C', A and B must lie on the plane axis[1] of the circle whose axial coordinates are

$$\pi_{ij}=\begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix}, \quad (i,j=0,\ 1,\ 2,\ 3\ ;\ i\neq j).$$

Let (ξ) and (η) be the coordinates of the vertices A and B respectively, and θ the angle between the two lines CA and CC', then

$$a'_i=\xi_i\cos\theta+\eta_i\sin\theta, \quad (i=0,\ 1,\ 2,\ 3) \tag{3}$$

and

$$\lambda b_i=\frac{\partial}{\partial r_i}\begin{vmatrix} r_0 & r_1 & r_2 & r_3 \\ a_0 & a_1 & a_2 & a_3 \\ \xi_0 & \xi_1 & \xi_2 & \xi_3 \\ \eta_0 & \eta_1 & \eta_2 & \eta_3 \end{vmatrix},$$

(1) See T. Nishiuchi, 'Oriented circles in Non-Euclidean Space,' loc. cit.

$$(\bar{\xi}\bar{\xi})=(\eta\eta)=k^2,$$

$$(a\bar{\xi})=(a\eta)=(\bar{\xi}\eta)=0,$$

where λ is a proportional factor.

Then, the coordinates of a point on the circle given by (2) will be

$$\begin{aligned}
x_0&=a_0\cos\frac{R}{k}+(\bar{\xi}_0\cos\theta+\eta_0\sin\theta)\sin\frac{R}{k},\\
x_1&=a_1\cos\frac{R}{k}+(\bar{\xi}_1\cos\theta+\eta_1\sin\theta)\sin\frac{R}{k},\\
x_2&=a_2\cos\frac{R}{k}+(\bar{\xi}_2\cos\theta+\eta_2\sin\theta)\sin\frac{R}{k},\\
x_3&=a_3\cos\frac{R}{k}+(\bar{\xi}_3\cos\theta+\eta_3\sin\theta)\sin\frac{R}{k},
\end{aligned}\tag{4}$$

where

$$(aa)=(\bar{\xi}\bar{\xi})=(\eta\eta)=k^2,$$

$$(a\bar{\xi})=(a\eta)=(\bar{\xi}\eta)=0.$$

§ 2. Focal Point.

Consider a congruence of circles defined by the equations

$$x_i=a_i\cos\frac{R}{k}+(\bar{\xi}_i\cos\theta+\eta_i\sin\theta)\sin\frac{R}{k},\tag{5}$$

$$(i=0,\ 1,\ 2,\ 3),$$

where a_i, $\bar{\xi}_i$, η_i $(i=0,\ 1,\ 2,\ 3)$, R are analytic functions of the two parameters u and v, and θ is a variable along the circle.

The parameters u and v determine the circle and θ a point upon the circle.

If we establish a relation between u and v, say

$$v=f(u)$$

the circles of the congruence whose parameters satisfy this relation depend upon the single parameter u and consequently form a surface; the parametric equations of this surface are given by (5), when v has

been replaced by $f(u)$. The surface will evidently be changed according as the function $f(u)$ changes, and are called the *surface of the congruence* by Prof. Darboux.[1]

The equation of the tangent plane to the surface of the congruence is

$$(6)\quad \begin{vmatrix} X_0 & X_1 & X_2 & X_3 \\ x_0 & x_1 & x_2 & x_3 \\ \dfrac{\partial x_0}{\partial \theta} & \dfrac{\partial x_1}{\partial \theta} & \dfrac{\partial x_2}{\partial \theta} & \dfrac{\partial x_3}{\partial \theta} \\ \dfrac{\partial x_0}{\partial u}+\dfrac{\partial x_0}{\partial v}f'(u) & \dfrac{\partial x_1}{\partial u}+\dfrac{\partial x_1}{\partial v}f'(u) & \dfrac{\partial x_2}{\partial u}+\dfrac{\partial x_2}{\partial v}f'(u) & \dfrac{\partial x_3}{\partial u}+\dfrac{\partial x_3}{\partial v}f'(u) \end{vmatrix} = 0,$$

where X_i $(i=0, 1, 2, 3)$ are current coordinates and the accent denotes differentiation. And we easily have the following theorem of Darboux:[2]

Theorem. For any four surfaces of a congruence containing the same circle of the congruence, the anharmonic ratio of the tangent planes to these surfaces at a point is constant when the point moves along the circle.

From (6) it is seen that when (x) satisfies the condition

$$(7)\quad \frac{\begin{vmatrix} x_0 & x_1 & x_2 \\ \dfrac{\partial x_0}{\partial \theta} & \dfrac{\partial x_1}{\partial \theta} & \dfrac{\partial x_2}{\partial \theta} \\ \dfrac{\partial x_0}{\partial u} & \dfrac{\partial x_1}{\partial u} & \dfrac{\partial x_2}{\partial u} \end{vmatrix}}{\begin{vmatrix} x_0 & x_1 & x_2 \\ \dfrac{\partial x_0}{\partial \theta} & \dfrac{\partial x_1}{\partial \theta} & \dfrac{\partial x_2}{\partial \theta} \\ \dfrac{\partial x_0}{\partial v} & \dfrac{\partial x_1}{\partial v} & \dfrac{\partial x_2}{\partial v} \end{vmatrix}} = \frac{\begin{vmatrix} x_0 & x_2 & x_3 \\ \dfrac{\partial x_0}{\partial \theta} & \dfrac{\partial x_2}{\partial \theta} & \dfrac{\partial x_3}{\partial \theta} \\ \dfrac{\partial x_0}{\partial u} & \dfrac{\partial x_2}{\partial u} & \dfrac{\partial x_3}{\partial u} \end{vmatrix}}{\begin{vmatrix} x_0 & x_2 & x_3 \\ \dfrac{\partial x_0}{\partial \theta} & \dfrac{\partial x_2}{\partial \theta} & \dfrac{\partial x_3}{\partial \theta} \\ \dfrac{\partial x_0}{\partial v} & \dfrac{\partial x_2}{\partial v} & \dfrac{\partial x_3}{\partial v} \end{vmatrix}} = \cdots\cdots C \text{ (const.)},$$

the equation of the tangent plane is independent of the function $f(u)$ and consequently is the same for all surfaces of the congruence through

(1), (2) See Darboux, Leçons sur la théorie générale des surfaces, Vol. ii.

the corresponding point. These points are called the *focal point* and their locus the *focal surface* by Prof. Darboux.[1]

The equation (7) may be represented by

$$\begin{vmatrix} x_0 & x_1 & x_2 & x_3 \\ \frac{\partial x_0}{\partial \theta} & \frac{\partial x_1}{\partial \theta} & \frac{\partial x_2}{\partial \theta} & \frac{\partial x_3}{\partial \theta} \\ \frac{\partial x_0}{\partial u} & \frac{\partial x_1}{\partial u} & \frac{\partial x_2}{\partial u} & \frac{\partial x_3}{\partial u} \\ \frac{\partial x_0}{\partial v} & \frac{\partial x_1}{\partial v} & \frac{\partial x_2}{\partial v} & \frac{\partial x_3}{\partial v} \end{vmatrix} = 0,$$

from which we have the relation

$$\frac{\partial x_i}{\partial \theta} = \lambda x_i + \mu \frac{\partial x_i}{\partial u} + \nu \frac{\partial x_i}{\partial v}, \quad (i = 0, 1, 2, 3),$$

where λ, μ and ν are a parameter respectively.

Multiplying through by x_i and adding, we get

$$\lambda = 0,$$

and have the following relation

$$(8) \qquad \frac{\partial x_i}{\partial \theta} = \mu \frac{\partial x_i}{\partial u} + \nu \frac{\partial x_i}{\partial v}, \quad (i = 0, 1, 2, 3).$$

Again, multiplying through by a_i and adding, we get

$$(9) \qquad \mu\left[\left(a\frac{\partial a'}{\partial u}\right) - k\frac{\partial R}{\partial u}\right] + \nu\left[\left(a\frac{\partial a'}{\partial v}\right) - k\frac{\partial R}{\partial v}\right] = 0,$$

and the relation (8) will be

$$(10) \qquad k^2 \sin\frac{R}{k}\frac{\partial a_i'}{\partial \theta} = \mu\left[k^2\left(\frac{\partial a_i}{\partial u}\cos\frac{R}{k} + \frac{\partial a_i'}{\partial u}\sin\frac{R}{k}\right)\right.$$

$$\left. - \left(a_i \sin\frac{R}{k} - a_i' \cos\frac{R}{k}\right)\left(a\frac{\partial a'}{\partial u}\right)\right] +$$

(1) See Darboux, Leçons sur la théorie générale des surfaces, Vol. ii.

$$+\nu\left[k^2\left(\frac{\partial a_i}{\partial v}\cos\frac{R}{k}+\frac{\partial a_i'}{\partial v}\sin\frac{R}{k}\right)-\left(a_i\sin\frac{R}{k}-a_i'\cos\frac{R}{k}\right)\left(a\frac{\partial a'}{\partial v}\right)\right].$$

$$(i=0,\ 1,\ 2,\ 3).$$

Here we adopt the following notation in analogy to Kummer's classical method[1]

$$k^2(dada)-(a'da)=\begin{vmatrix} a_0' & a_1' & a_2' & a_3' \\ da_0 & da_1 & da_2 & da_3 \end{vmatrix}^2$$

$$=E_{11}du^2+(E_{12}+E_{21})dudv+E_{22}dv^2,\ (E_{12}=E_{21})$$

$$k^2(da'da')-(ada')^2=\begin{Vmatrix} a_0 & a_1 & a_2 & a_3 \\ da_0' & da_1' & da_2' & da_3' \end{Vmatrix}^2$$

$$(11)\quad =E_{11}'du^2+E_{22}'dv^2+E_{33}'d\theta^2+(E_{23}'+E'_{32})dvd\theta$$

$$+(E_{31}'+E_{13}')d\theta du+(E_{12}'+E_{21}')dudv,\ (E_{ij}=E_{ji},\ i,j=1,2,3)$$

$$k^2(dada')=\begin{Vmatrix} a_0 & a_1 & a_2 & a_3 \\ da_0 & da_1 & da_2 & da_3 \end{Vmatrix}\cdot\begin{Vmatrix} a_0 & a_1 & a_2 & a_3 \\ da_0' & da_1' & da_2' & da_3' \end{Vmatrix}$$

$$=\begin{Vmatrix} a_0' & a_1' & a_2' & a_3' \\ da_0' & da_1' & da_2' & da_3' \end{Vmatrix}\cdot\begin{Vmatrix} a_0' & a_1' & a_2' & a_3' \\ da_0 & da_1 & da_2 & da_3 \end{Vmatrix}$$

$$=e_{11}du^2+e_{22}dv^2+e_{33}d\theta^2+(e_{23}+e_{32})dvd\theta$$

$$+(e_{31}+e_{13})d\theta du+(e_{12}+e_{21})dudv,$$

$$(e_{ij}\neq e_{ji},\ i,j=1,2,3,\ i\neq j)$$

$$(12)\quad \left|E_{ij}\right|=\left|a'\ a\ \frac{\partial a}{\partial u}\ \frac{\partial a}{\partial v}\right|^2\equiv\triangle^2,$$

$$\left|E'_{ij}\right|=k^2\left|a\ a'\ \frac{\partial a'}{\partial u}\ \frac{\partial a'}{dv}\ \frac{\partial a'}{\partial\theta}\right|^2\equiv\triangle'^2,$$

$$\triangle^2\triangle'^2=|e_{ij}|^2.$$

---

(1) See 'Allgemeine Theorie der geradlinigen Strahlensysteme,' Crelles Journal LVII, p. 189–230, 1860.

Multiplying through (10) by $\frac{\partial a_i'}{\partial u}$, $\frac{\partial a_i'}{dv}$ and $\frac{\partial a_i'}{\partial \theta}$ respectively and adding

$$\sin\frac{R}{k}E_{13}' = \mu\left[e_{11}\cos\frac{R}{k} + E_{11}'\sin\frac{R}{k}\right] + \nu\left[e_{21}\cos\frac{R}{k} + E_{21}'\sin\frac{R}{k}\right],$$

$$(13)\quad \sin\frac{R}{k}E_{23}' = \mu\left[e_{12}\cos\frac{R}{k} + E_{12}'\sin\frac{R}{k}\right] + \nu\left[e_{22}\cos\frac{R}{k} + E_{22}'\sin\frac{R}{k}\right],$$

$$\sin\frac{R}{k}E_{33}' = \mu\left[e_{13}\cos\frac{R}{k} + E_{13}'\sin\frac{R}{k}\right] + \nu\left[e_{23}\cos\frac{R}{k} + E_{23}'\sin\frac{R}{k}\right].$$

Similarly, we have

$$\sin\frac{R}{k}e_{13} = \mu\left[E_{11}\cos\frac{R}{k} + e_{11}\sin\frac{R}{k}\right] + \nu\left[E_{12}\cos\frac{R}{k} + e_{12}\sin\frac{R}{k}\right],$$

$$(14)$$

$$\sin\frac{R}{k}e_{23} = \mu\left[E_{21}\cos\frac{R}{k} + e_{21}\sin\frac{R}{k}\right] + \nu\left[E_{22}\cos\frac{R}{k} + e_{22}\sin\frac{R}{k}\right].$$

From the above two equations, we can find the value of μ and ν. Eliminating μ and ν from (13), we get

$$(15)\quad \begin{vmatrix} e_{11}\cos\frac{R}{k} + E_{11}'\sin\frac{R}{k} & e_{21}\cos\frac{R}{k} + E_{21}'\sin\frac{R}{k} & E_{31}'\sin\frac{R}{k} \\ e_{12}\cos\frac{R}{k} + E_{12}'\sin\frac{R}{k} & e_{22}\cos\frac{R}{k} + E_{22}'\sin\frac{R}{k} & E_{32}'\sin\frac{R}{k} \\ e_{13}\cos\frac{R}{k} + E_{13}'\sin\frac{R}{k} & e_{23}\cos\frac{R}{k} + E_{23}'\sin\frac{R}{k} & E_{33}'\sin\frac{R}{k} \end{vmatrix} = 0.$$

This equation can be looked upon as an equation in θ having its coefficients functions of u and v. When u and v are given particular values of θ satisfying this equation define the focal points upon the corresponding circle. When this equation is solved for θ the various solutions are substituted in (5), the latter will define the sheets of focal surface.

When the fixed point is common to all the planes of the circles of the congruence, the equation for the focal point (taking the center of the circles as the fixed point) will be

$$(16)\qquad \begin{vmatrix} E_{11}' & E_{21}' & E_{31}' \\ E_{12}' & E_{22}' & E_{32}' \\ E_{13}' & E_{23}' & E_{33}' \end{vmatrix} = 0.$$

Next, multiplying through the equation (10) by b_i and adding we get

$$\mu\left[\left(b\frac{\partial a}{\partial u}\right)\cos\frac{R}{k}+\left(b\frac{\partial a'}{\partial u}\right)\sin\frac{R}{k}\right] + \nu\left[\left(b\frac{\partial a}{\partial v}\right)\cos\frac{R}{k}+\left(b\frac{\partial a'}{\partial v}\right)\sin\frac{R}{k}\right]=0.$$

Eliminating μ and ν from this equation and the equation (9) we get another form of equation for the focal points:

$$(17)\qquad \begin{vmatrix} \left(a'\frac{\partial a}{\partial u}\right)+k\frac{\partial R}{\partial u} & \left(a'\frac{\partial a}{\partial v}\right)+k\frac{\partial R}{\partial v} \\ \left(b\frac{\partial a}{\partial u}\right)\cos\frac{R}{k}+\left(b\frac{\partial a'}{\partial u}\right)\sin\frac{R}{k} & \left(b\frac{\partial a}{\partial v}\right)\cos\frac{R}{k}+\left(b\frac{\partial a'}{\partial v}\right)\sin\frac{R}{k} \end{vmatrix}=0,$$

or

$$(18)\qquad \begin{vmatrix} \left(\xi\frac{\partial a}{\partial u}\right)\cos\theta+\left(\eta\frac{\partial a}{\partial u}\right)\sin\theta+k\frac{\partial R}{\partial u} & \left(\xi\frac{\partial a}{\partial v}\right)\cos\theta+\left(\eta\frac{\partial a}{\partial v}\right)\sin\theta+k\frac{\partial R}{\partial v} \\ \left(b\frac{\partial\xi}{\partial u}\right)\cos\theta+\left(b\frac{\partial\eta}{\partial u}\right)\sin\theta+\left(b\frac{\partial a}{\partial u}\right)\cos\frac{R}{k} & \left(b\frac{\partial\xi}{\partial v}\right)\cos\theta+\left(b\frac{\partial\eta}{\partial v}\right)\sin\theta+\left(b\frac{\partial a}{\partial v}\right)\cos\frac{R}{k} \end{vmatrix}=0.$$

This equation leads to four values of θ in general, which settle the focal points of the circle, so that there are four focal points upon a circle in a general congruence.

When the planes of the circles all pass through a fixed point, taking it as the center of the circles, then the equation for the focal points will be

$$\left[\left(b\frac{\partial\xi}{\partial u}\right)\frac{\partial R}{\partial v}-\left(b\frac{\partial\xi}{\partial v}\right)\frac{\partial R}{\partial u}\right]\cos\theta$$

$$+\left[\left(b\frac{\partial\eta}{\partial u}\right)\frac{\partial R}{\partial v}-\left(b\frac{\partial u}{\partial v}\right)\frac{\partial R}{\partial u}\right]\sin\theta=0.$$

And this equation leads to two values of θ determing two focal points upon each such a circle.

From the results above we get the following theorem:

Theorem. Upon a circle of a general congruence there are four focal points at which all the surfaces of the congruence through the circle admit the same tangent plane, whatever be the law according to which the circles have been assembled to generate the surfaces.

§ 3. Some Properties of the Focal Point.

We know that, in rectilinear congruences, certain selected consecutive rays intersect one another; the points of intersection of any ray with the different rays which meet it, are the focal points of the ray.[1] It is natural to enquire which circle (if any), consecutive to a given circle, do intersect. Now we shall find such circles.

Any circle

$$x_i=a_i\cos\frac{R}{k}+a_i'\sin\frac{R}{k},$$

$$a_i'=\xi_i\cos\theta+\eta_i\sin\theta,\ (i=0,\ 1,\ 2,\ 3)$$

intersects with a consecutive circle, if

$$a_i\cos\frac{R}{k}+a_i'\sin\frac{R}{k}=(a_i+da_i)\cos\frac{R+dR}{k}+(a_i'+da_i')\sin\frac{R+dR}{k},$$

$$(i=0,\ 1,\ 2,\ 3)$$

or

$$(19)\qquad da_i\cos\frac{R}{k}+da_i'\sin\frac{R}{k}-\frac{1}{k}\left(a_i\sin\frac{R}{k}-a_i'\cos\frac{R}{k}\right)dR=0,$$

$$(i=0,\ 1,\ 2,\ 3).$$

Multiplying through by a_i, and adding

---

1 See Fibbi, 'I sistemi doppiamente infiniti di raggi negli spazii di curvatura costante,' Annali della R. Scuola Normale Superiore, Pisa, 1891. Also, see Coolidge, Non-Euclidean Geometry.

$$(ada')=kdR. \tag{20}$$

Rewriting the equation (19)

$$\begin{aligned}&\left[k^2\left(\frac{\partial a_i}{\partial u}\cos\frac{R}{k}+\frac{\partial a_i'}{\partial u}\sin\frac{R}{k}\right)\right.\\&\left.-\left(a_i\sin\frac{R}{k}-a_i'\cos\frac{R}{k}\right)\left(a\frac{\partial a'}{\partial u}\right)\right]du\\&+\left[k^2\left(\frac{\partial a_i}{\partial v}\cos\frac{R}{k}+\frac{\partial a_i'}{\partial v}\sin\frac{R}{k}\right)\right.\\&\left.-\left(a_i\sin\frac{R}{k}-a_i'\cos\frac{R}{k}\right)\left(a\frac{\partial a'}{\partial v}\right)\right]dv\\&+\left[k^2\frac{\partial a_i'}{\partial\theta}\sin\frac{R}{k}-\left(a_i\sin\frac{R}{k}-a_i'\cos\frac{R}{k}\right)\left(a\frac{\partial a'}{\partial\theta}\right)\right]d\theta=0,\end{aligned} \tag{21}$$

$$(i=0,\ 1,\ 2,\ 3).$$

Multiplying through by $\frac{\partial a_i'}{\partial u}$, $\frac{\partial a_i'}{\partial v}$, $\frac{\partial a_i'}{\partial\theta}$ and adding respectively, we get

$$\begin{aligned}&\left[e_{11}\cos\frac{R}{k}+E_{11}'\sin\frac{R}{k}\right]du+\left[e_{21}\cos\frac{R}{k}+E_{21}'\sin\frac{R}{k}\right]dv\\&+E_{31}'\sin\frac{R}{k}d\theta=0,\\&\left[e_{12}\cos\frac{R}{k}+E_{12}'\sin\frac{R}{k}\right]du+\left[e_{22}\cos\frac{R}{k}+E_{22}'\sin\frac{R}{k}\right]dv\\&+E_{32}'\sin\frac{R}{k}d\theta=0,\\&\left[e_{13}\cos\frac{R}{k}+E_{13}'\sin\frac{R}{k}\right]du+\left[e_{23}\cos\frac{R}{k}+E_{23}'\sin\frac{R}{k}\right]dv\\&+E_{33}'\sin\frac{R}{k}d\theta=0.\end{aligned} \tag{22}$$

Eliminating $du:dv:d\theta$, we have

$$\begin{vmatrix} e_{11}\cos\frac{R}{k}+E_{11}'\sin\frac{R}{k} & e_{21}\cos\frac{R}{k}+E_{21}'\sin\frac{R}{k} & E_{31}'\sin\frac{R}{k} \\ e_{12}\cos\frac{R}{k}+E_{12}'\sin\frac{R}{k} & e_{22}\cos\frac{R}{k}+E_{22}'\sin\frac{R}{k} & E_{32}'\sin\frac{R}{k} \\ e_{13}\cos\frac{R}{k}+E_{13}'\sin\frac{R}{k} & e_{23}\cos\frac{R}{k}+E_{23}'\sin\frac{R}{k} & E_{33}'\sin\frac{R}{k} \end{vmatrix}=0,$$

which is the same equation as (15) giving the focal points of the circle. Hence we have the following theorem:

Theorem. The points of intersection of a circle with the consecutive circles are its focal points.

Again, multiplying through (21) by b_i and adding, we get

$$\left[\left(b\frac{\partial a}{\partial u}\right)\cos\frac{R}{k}+\left(b\frac{\partial a'}{\partial u}\right)\sin\frac{R}{k}\right]du$$

$$+\left[\left(b\frac{\partial a}{\partial v}\right)\cos\frac{R}{k}+\left(b\frac{\partial a'}{\partial v}\right)\sin\frac{R}{k}\right]dv=0,$$

and from (20)

$$\left[\left(a\frac{\partial a'}{\partial u}\right)-k\frac{\partial R}{\partial u}\right]du+\left[\left(a\frac{\partial a'}{\partial v}\right)-k\frac{\partial R}{\partial v}\right]dv=0.$$

Eliminating $du:dv$ from the above two equations, we get

$$\begin{vmatrix} \left(a\frac{\partial a'}{\partial u}\right)-k\frac{\partial R}{\partial u} & \left(a\frac{\partial a'}{\partial v}\right)-k\frac{\partial R}{\partial v} \\ \left(b\frac{\partial a}{\partial u}\right)\cos\frac{R}{k}+\left(b\frac{\partial a'}{\partial u}\right)\sin\frac{R}{k} & \left(b\frac{\partial a}{\partial v}\right)\cos\frac{R}{k}+\left(b\frac{\partial a'}{\partial v}\right)\sin\frac{R}{k} \end{vmatrix}=0,$$

and this is the same equation as (17) which gives the facal points on the circle.

Rewriting the above two equations, we have

$$\cos\frac{R}{k}\left[\left(b\frac{\partial a}{\partial u}\right)du+\left(b\frac{\partial a}{\partial v}\right)dv\right]+\sin\frac{R}{k}\left[\left(b\frac{\partial\xi}{\partial u}\right)du+\left(b\frac{\partial\xi}{\partial v}\right)dv\right]\cos\theta$$

$$+\sin\frac{R}{k}\left[\left(b\frac{\partial\eta}{\partial u}\right)du+\left(b\frac{\partial\eta}{\partial v}\right)dv\right]\sin\theta=0,$$

(23)

$$k\left(\frac{\partial R}{\partial u}du+\frac{\partial R}{\partial v}dv\right)-\left[\left(a\frac{\partial\xi}{\partial u}\right)du+\left(a\frac{\partial\xi}{\partial v}\right)dv\right]\cos\theta$$

$$-\left[\left(a\frac{\partial\eta}{\partial u}\right)du+\left(a\frac{\partial\eta}{\partial v}\right)dv\right]\sin\theta=0.$$

Eliminating $\cos\theta$ and $\sin\theta$, we get

$$\sin^2\frac{R}{k}\begin{vmatrix}\left(b\frac{\partial\xi}{\partial u}\right)du+\left(b\frac{\partial\xi}{\partial v}\right)dv & \left(b\frac{\partial\eta}{\partial u}\right)du+\left(b\frac{\partial\eta}{\partial v}\right)dv\\ \left(a\frac{\partial\xi}{\partial u}\right)du+\left(a\frac{\partial\xi}{\partial v}\right)dv & \left(a\frac{\partial\eta}{\partial u}\right)du+\left(a\frac{\partial\eta}{\partial v}\right)dv\end{vmatrix}^2$$

$$=\begin{vmatrix}\sin\frac{R}{k}\left[\left(b\frac{\partial\xi}{\partial u}\right)du+\left(b\frac{\partial\xi}{\partial v}\right)dv\right] & \cos\frac{R}{k}\left[\left(b\frac{\partial a}{\partial u}\right)du+\left(b\frac{\partial a}{\partial v}\right)dv\right]\\ \left(a\frac{\partial\xi}{\partial u}\right)du+\left(a\frac{\partial\xi}{\partial v}\right)dv & -k\left(\frac{\partial R}{\partial u}du+\frac{\partial R}{\partial v}dv\right)\end{vmatrix}^2$$

(24)

$$+\begin{vmatrix}\sin\frac{R}{k}\left[\left(b\frac{\partial\eta}{\partial u}\right)du+\left(b\frac{\partial\eta}{\partial v}\right)dv\right] & \cos\frac{R}{k}\left[\left(b\frac{\partial a}{\partial u}\right)du+\left(b\frac{\partial a}{\partial v}\right)dv\right]\\ \left(a\frac{\partial\eta}{\partial u}\right)du+\left(a\frac{d\eta}{\partial v}\right)dv & -k\left(\frac{\partial R}{\partial u}du+\frac{\partial R}{\partial v}dv\right)\end{vmatrix}^2$$

The circles, that are consecutive and intersect, are determined by quantities $u+du$, $v+dv$; and the common point with the consecutive circle is given by the value of $\theta+d\theta$ on that consecutive circle. Now the above relation is an equation for $du:dv$; its coefficient are functions of u and v only; and therefore it determines four consecutive values of u and v, which give consecutive values of a_i, b_i ($i=0, 1, 2, 3$) and R, and therefore give fous consecutive intersecting circles,

Theorem. In a general congruence of circles, every circle meets four other consecutive circles; it intersects each of the circles at a single point, the four points being the focal points of the circle.

Two circles in space can not intersect at more than two points. So we can consider such a congruence of circles that two of the four focal points of a circle shall lie on one consecutive circle, and the other two lie on another consecutive circle. In this case, there will be two (and not four) values of $du:dv$ which give two consecutive intersectlng circles; for each of these two values of $du:dv$, there will be two values of θ, giving the two points upon the consecutive circle which are focal points of the first. In order that this may be possible, the two equations

$$\cos\frac{R}{k}(bda)+\sin\frac{R}{k}(bd\xi)\cos\theta+\sin\frac{R}{k}(bd\eta)\sin\theta=0,$$

$$kdR-(ad\xi)\cos\theta-(ad\eta)\sin\theta=0,$$

can not be independent; for if not, they would determine $\cos\theta$ and $\sin\theta$ uniquely for an assigned value of $du:dv$, and so there would be only a single focal point on the consecutive circle. Hence, all the determinants of the matrix

$$(25)\qquad \begin{Vmatrix} (bda) & \tan\frac{R}{k}(bd\xi) & \tan\frac{R}{k}(bd\eta) \\ -kdR & (ad\xi) & (ad\eta) \end{Vmatrix}$$

must be zero, and these equations are to determine two values of θ.

But, since, if two circles (consecutive) intersect at two points they are cospherical and the conditions for that is

$$\begin{vmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \\ da_0 & da_1 & da_2 & da_3 \\ db_0 & db_1 & db_2 & db_3 \end{vmatrix}=0.$$

and all the determinants of the matrix

$$\begin{Vmatrix} a_0\sin\frac{R}{k}\frac{dR}{k}+da_0\cos\frac{R}{k} & a_1\sin\frac{R}{k}\frac{dR}{k}+da_1\cos\frac{R}{k} \\ b_0 & b_1 \\ db_0 & db_1 \\ a_2\sin\frac{R}{k}\frac{dR}{k}+da_2\cos\frac{R}{k} & a_3\sin\frac{R}{k}\frac{dR}{k}+da_3\cos\frac{R}{k} \\ b_2 & b_3 \\ db_2 & db_3 \end{Vmatrix}$$

are equal to zero.[1] From these relations we can also find the conditions (25) easily.

---

1 See, T. Nishiuchi, 'Oriented Circles in Non-Euclidean Space,' loc. cit.

Next, from the conditions (25) we can find two quantities λ and μ such that

$$\tan\frac{R}{k}\left[\lambda\left(b\frac{\partial\xi}{\partial u}\right)+\mu\left(b\frac{\partial\eta}{\partial u}\right)\right]=\left(b\frac{\partial a}{\partial u}\right),$$

$$\tan\frac{R}{k}\left[\lambda\left(b\frac{\partial\xi}{\partial v}\right)+\mu\left(b\frac{\partial\eta}{\partial v}\right)\right]=\left(b\frac{\partial a}{\partial v}\right),$$

$$\lambda\left(a\frac{\partial\xi}{\partial u}\right)+\mu\left(a\frac{\partial\eta}{\partial u}\right)=k\frac{\partial R}{\partial u},$$

$$\lambda\left(a\frac{\partial\xi}{\partial v}\right)+\mu\left(a\frac{\partial\eta}{\partial v}\right)=k\frac{\partial R}{\partial v}.$$

Therefore, the two conditions represented by the following equations

$$\begin{Vmatrix} \left(a\frac{\partial\xi}{\partial u}\right) & \left(a\frac{\partial\xi}{\partial v}\right) & \left(b\frac{\partial\xi}{\partial u}\right) & \left(b\frac{\partial\xi}{\partial v}\right) \\ \left(a\frac{\partial\eta}{\partial u}\right) & \left(a\frac{\partial\eta}{\partial v}\right) & \left(b\frac{\partial\eta}{\partial u}\right) & \left(b\frac{\partial\eta}{\partial v}\right) \\ k\frac{\partial R}{\partial u} & k\frac{\partial R}{\partial v} & \left(b\frac{\partial a}{\partial u}\right) & \left(b\frac{\partial a}{\partial v}\right) \end{Vmatrix}=0,$$

must be satisfied by the magnitudes that occur in the expression of the congruence. The two values of $du:dv$ are the roots of the quadratic

$$\begin{vmatrix} \left(a\frac{\partial\xi}{\partial u}\right)du+\left(a\frac{\partial\xi}{\partial v}\right)dv & \left(a\frac{\partial\eta}{\partial u}\right)du+\left(a\frac{\partial\eta}{\partial v}\right)dv \\ \left(b\frac{\partial\xi}{\partial u}\right)du+\left(b\frac{\partial\xi}{\partial v}\right)dv & \left(b\frac{\partial\eta}{du}\right)du+\left(b\frac{\partial\eta}{\partial v}\right)dv \end{vmatrix}=0.$$

And the two values of θ, which correspond to one of the values of $du:dv$ are the roots of the equation

$$k\left(\frac{\partial R}{\partial u}du+\frac{\partial R}{\partial v}dv\right)+\left[\left(a\frac{\partial\xi}{\partial u}\right)du+\left(a\frac{\partial\xi}{\partial v}\right)dv\right]\cos\theta$$

$$+\left[\left(a\frac{\partial\eta}{\partial u}\right)du+\left(a\frac{\partial\eta}{\partial v}\right)dv\right]\sin\theta=0.$$

CHAPTER. II.

CYCLICAL SYSTEMS OF CONGRUENCES OF CIRCLES.

§ 4. Condition that a Congruence of Circles has an Orthogonal Surface.

The Plücker line-coordinates of the tangent to any circle of a congruence represented by

$$x_i = a_i \cos\frac{R}{k} + (\xi_i \cos\theta + \eta_i \sin\theta)\sin\frac{R}{k}, \tag{1}$$

$$(i = 0.\ 1,\ 2,\ 3)$$

at a point (x) are given by

$$\begin{vmatrix} x_i & x_j \\ \dfrac{\partial x_i}{\partial\theta} & \dfrac{\partial x_j}{\partial\theta} \end{vmatrix}, \quad (i, j = 0,\ 1,\ 2,\ 3\ ;\ i \neq j).$$

Hence if every line at that point whose Plücker line coordinates are

$$\begin{vmatrix} x_i & x_j \\ dx_i & dx_j \end{vmatrix}, \quad (i, j = 0,\ 1,\ 2,\ 3\ ;\ i \neq j)$$

perpendicular to the tangent to the circle, must satisfy the relation

$$\begin{Vmatrix} x_0 & x_1 & x_2 & x_3 \\ \dfrac{\partial x_0}{\partial\theta} & \dfrac{\partial x_1}{\partial\theta} & \dfrac{\partial x_2}{\partial\theta} & \dfrac{\partial x_3}{\partial\theta} \end{Vmatrix} \cdot \begin{Vmatrix} x_0 & x_1 & x_2 & x_3 \\ dx_0 & dx_1 & dx_2 & dx_3 \end{Vmatrix} = 0, \tag{2}$$

i. e.

$$\left(\frac{\partial x}{\partial\theta} dx\right) = 0. \tag{3}$$

Now, we put

$$U = \frac{1}{\sin\dfrac{R}{k}}\left(\frac{\partial x}{\partial\theta}\ \frac{\partial x}{\partial u}\right)$$

$$= -\left(\xi\frac{\partial a}{\partial u}\right)\cos\frac{R}{k}\sin\theta + \left(\eta\frac{\partial a}{\partial u}\right)\cos\frac{R}{k}\cos\theta + \left(\eta\frac{\partial\xi}{\partial u}\right)\sin\frac{R}{k},$$

$$(4)\quad V=\frac{1}{\sin\frac{R}{k}}\left(\frac{\partial x}{\partial\theta}\,\frac{\partial x}{\partial v}\right)$$

$$=-\left(\xi\frac{\partial a}{\partial v}\right)\cos\frac{R}{k}\sin\theta+\left(\eta\frac{\partial a}{\partial v}\right)\cos\frac{R}{k}\cos\theta+\left(\eta\frac{\partial\xi}{\partial v}\right)\sin\frac{R}{k},$$

$$\Theta=\frac{1}{\sin\frac{R}{k}}\left(\frac{\partial x}{\partial\theta}\,\frac{\partial x}{\partial\theta}\right)=k^2\sin\frac{R}{k},$$

then the equation (3) will be

$$(5)\qquad U du+Vdv+\Theta d\theta=0.$$

And the necessary and sufficient condition that there is a system of surface which cuts the circle of the congruence orthogonally, is that the differential equation (5) shall be integrable, i. e. for all values of u, v and θ, the condition of integrability

$$(6)\quad \Theta\left[\frac{\partial U}{\partial v}-\frac{\partial V}{\partial u}\right]+U\left[\frac{\partial V}{\partial\theta}-\frac{\partial\Theta}{\partial v}\right]+V\left[\frac{\partial\Theta}{\partial u}-\frac{\partial U}{\partial\theta}\right]=0,$$

must satisfy. When the values of U, V and Θ are substituted, the condition will be

$$(7)\qquad A+B\sin\theta+C\cos\theta=0.$$

where

$$A=\cos^2\frac{R}{k}\left[\left(\xi\frac{\partial a}{\partial u}\right)\left(\eta\frac{\partial a}{\partial v}\right)-\left(\eta\frac{\partial a}{\partial u}\right)\left(\xi\frac{\partial a}{\partial v}\right)\right]$$

$$+k^2\sin^2\frac{R}{k}\left[\frac{\partial}{\partial v}\left(\eta\frac{\partial\xi}{\partial u}\right)-\frac{\partial}{\partial u}\left(\eta\frac{\partial\xi}{\partial v}\right)\right],$$

$$(8)\quad B=\sin\frac{R}{k}\cos\frac{R}{k}\left[\left(\eta\frac{\partial a}{\partial u}\right)\left(\eta\frac{\partial\xi}{\partial v}\right)-\left(\eta\frac{\partial a}{\partial v}\right)\left(\eta\frac{\partial\xi}{\partial u}\right)\right.$$

$$\left.+k^2\left\{\frac{\partial}{\partial u}\left(\xi\frac{\partial a}{\partial v}\right)-\frac{\partial}{\partial v}\left(\xi\frac{\partial a}{\partial u}\right)\right\}\right]+k\left[\left(\xi\frac{\partial a}{\partial u}\right)\frac{\partial R}{\partial v}-\left(\xi\frac{\partial a}{\partial v}\right)\frac{\partial R}{\partial u}\right],$$

$$C=\sin\frac{R}{k}\cos\frac{R}{k}\left[\left(\xi\frac{\partial a}{\partial u}\right)\left(\eta\frac{\partial\xi}{\partial v}\right)-\left(\xi\frac{\partial a}{\partial v}\right)\left(\eta\frac{\partial\xi}{\partial u}\right)\right.$$

$$\left.+k^2\left\{\frac{\partial}{\partial v}\left(\eta\frac{\partial a}{\partial u}\right)-\frac{\partial}{\partial u}\left(\eta\frac{\partial a}{\partial v}\right)\right\}\right]+k\left[\left(\eta\frac{\partial a}{\partial v}\right)\frac{\partial R}{\partial u}-\left(\eta\frac{\partial a}{\partial u}\right)\frac{\partial R}{\partial v}\right],$$

and A, B and C are manifestly independent of θ.

If the condition be satisfied identically, then

$$A=0, \quad B=0, \quad C=0.$$

In this case, let

$$\Phi=\text{const.}$$

be the integral solution of the differential equation; it is the equation of the family of surfaces cutting the circles of the congruence orthogonally. If the condition is not satisfied identically, it is possible that the two solutions of the equation (7) will satisfy (5), and thus determine two surfaces orthogonal to the circles of the congruence. Hence we have the theorem of Ribaucour[1]:

Theorem. If the circle of a congruence are normal to more than two surfaces, they form a cyclical system.

§ 5. A Special Congruence of Circles Which From a Cyclical System.

Consider a congruence of circles, which lie in the tangent planes to a surface and have their centers at the point of contact of the tangent plane with the surface.

Let the surface be referred to the lines of curvature as the parametric curves; then we can take the points (ξ) and (η) on the tangents to the parametric curves $v=\text{const}$, $u=\text{const.}$ respectively, so that

$$\xi_i=\lambda a_i+\mu\frac{\partial a_i}{\partial u}du,$$

$$\eta_i=\lambda' a_i+\mu'\frac{\partial a_i}{\partial v}dv, \qquad (i=0, 1, 2, 3)$$

where λ, μ and λ', μ' are parameters.

But since

$$(a\xi)=0, \quad (a\eta)=0,$$

$$(\xi\xi)=(\eta\eta)=k^2,$$

1 See Ribaucour, 'Mèmoire sur la théorie générale des surfaces courbées,' Journal des Mathématiques, Ser. 4, Vol. vii, 1891.

we have

$$\xi_i=\frac{k}{\sqrt{\left(\frac{\partial a}{\partial u}\,\frac{\partial a}{\partial u}\right)}}\frac{\partial a_i}{\partial u}=\frac{k}{\sqrt{E}}\frac{\partial x_i}{\partial u},$$

$$\eta_i=\frac{k}{\sqrt{\left(\frac{\partial a}{\partial v}\,\frac{\partial a}{\partial v}\right)}}\frac{\partial a_i}{\partial v}=\frac{k}{\sqrt{G}}\frac{\partial x_i}{\partial v},\quad (i=0,\ 1,\ 2,\ 3)$$

and

$$\left(\xi\frac{\partial a}{\partial u}\right)=k\sqrt{E},\quad \left(\xi\frac{\partial a}{\partial v}\right)=0,$$

$$\left(\eta\frac{\partial a}{\partial u}\right)=0,\qquad \left(\eta\frac{\partial a}{\partial v}\right)=k\sqrt{G},$$

$$\left(\eta\frac{\partial\xi}{\partial u}\right)=-\frac{k^2}{2\sqrt{EG}}\frac{\partial E}{\partial v},$$

$$\left(\eta\frac{\partial\xi}{\partial v}\right)=\frac{k^2}{2\sqrt{EG}}\frac{\partial G}{\partial v}.$$

Substituting these relations above to (8), we have

$$A=k^2\sqrt{EG}-k^2\sqrt{EG}\sin^2\frac{R}{k}\left[1+\frac{k^2}{2EG}\left\{\frac{\partial^2 G}{\partial u^2}+\frac{\partial^2 E}{\partial v^2}\right.\right.$$

$$\left.\left.-\frac{1}{2EG}\frac{\partial G}{\partial u}\left(G\frac{\partial E}{\partial u}+E\frac{\partial G}{\partial u}\right)-\frac{1}{2EG}\frac{\partial E}{\partial v}\left(G\frac{\partial E}{\partial v}+E\frac{\partial G}{\partial v}\right)\right\}\right],$$

$$B=k^2\sqrt{E}\frac{\partial R}{\partial v},$$

$$C=k^2\sqrt{G}\frac{\partial R}{\partial u}.$$

Let K be the total relative curvature of the surface, then

---

[1] E, F, G are fundamental coefficients of the first order, and

$$E=\left(\frac{\partial x}{\partial u}\,\frac{\partial x}{\partial u}\right),\quad F=\left(\frac{\partial x}{\partial u}\,\frac{\partial x}{\partial v}\right),\quad G=\left(\frac{\partial x}{\partial v}\,\frac{\partial x}{\partial v}\right).$$

But in this case, from the hypothesis, $(x)=(a)$ and

$$E=\left(\frac{\partial a}{\partial u}\,\frac{\partial a}{\partial u}\right),\quad F=\left(\frac{\partial a}{\partial u}\,\frac{\partial a}{\partial v}\right)=0,\quad G=\left(\frac{\partial a}{\partial v}\,\frac{\partial a}{\partial v}\right).$$

$$K=\frac{DD''^{1}}{k^2EG}=\frac{1}{k^2E^2G^2}\begin{vmatrix} k^2 & 0 & 0 & -G \\ 0 & E & 0 & -\frac{1}{2}\frac{\partial G}{\partial u} \\ 0 & 0 & G & \frac{1}{2}\frac{\partial G}{\partial v} \\ -E\frac{1}{2}\frac{\partial E}{\partial u} & -\frac{1}{2}\frac{\partial E}{\partial v} & \left(\frac{\partial^2 x}{\partial u^2}\ \frac{\partial^2 x}{\partial v^2}\right) & \end{vmatrix},$$

but

$$\left(\frac{\partial^2 x}{\partial u^2}\ \frac{\partial^2 x}{\partial v^2}\right)=\frac{1}{4EG}\left\{G\left(\frac{\partial E}{\partial v}\right)^2+E\left(\frac{\partial G}{\partial u}\right)^2\right\}-\frac{1}{2}\left\{\frac{\partial^2 E}{\partial v^2}+\frac{\partial^2 G}{\partial u^2}\right\},$$

we have

$$K=-\frac{1}{k^2}\left[1+\frac{k^2}{2EG}\left\{\frac{\partial^2 G}{\partial u^2}+\frac{\partial^2 E}{\partial v^2}-\frac{1}{2EG}\frac{\partial G}{\partial u}\left(G\frac{\partial E}{\partial u}+E\frac{\partial G}{\partial u}\right)\right.\right.$$

$$\left.\left.-\frac{1}{2EG}\frac{\partial E}{\partial v}\left(G\frac{\partial E}{\partial v}+E\frac{\partial G}{\partial v}\right)\right\}\right]$$

therefore

$$A=k^2\sqrt{EG}\left(1+k^2\sin^2\frac{R}{k}K\right).$$

In order that the congruence of the circles may form a cyclical system, we are to have

$$A=0,\quad B=0,\quad C=0.$$

Hence, we must have

$$R=\text{constant},$$

$$K=-\frac{1}{k^2\sin^2\dfrac{R}{k}}(\text{constant}),$$

---

1 D, D', D'' are fundamental coefficients of the second order, and

$$D=\frac{\left|x\ \frac{\partial x}{\partial u}\ \frac{\partial x}{\partial v}\ \frac{\partial^2 x}{\partial u^2}\right|}{\sqrt{EG-F^2}},\quad D'=\frac{\left|x\ \frac{\partial x}{\partial u}\ \frac{\partial x}{\partial v}\ \frac{\partial^2 x}{\partial u\partial v}\right|}{\sqrt{EG-F^2}},\quad D''=\frac{\left|x\ \frac{\partial x}{\partial u}\ \frac{\partial x}{\partial v}\ \frac{\partial^2 x}{\partial v^2}\right|}{\sqrt{EG-F^2}}.$$

and the total Gaussian curvature[1] of the surface will be constant and equal to $-\frac{1}{k^2 \tan^2 \frac{R}{k}}$.

Therefore, we get the following result:

A congruence of circle of constant radius which lie in the tangent planes to a surface, whose total Gaussian curvature is constant and its absolute valve is equal to the square of the curvature of the circles, and have their centers at the point of contact of the tangent plane with the surface, form a cyclical system.

§ 6. Formula for a Cyclical System.

Let S be an orthogonal surface of a cyclical system and take lines of curvature on S as parametric curves. Then through every point (x) on S, a circle (u, v), which is orthogonal to the surface S, passes.

Take points (X) and (Y) on the tangents to the parametric curves $v=$const. and $u=$const. respectively, such that

$$(Xx)=0,\ (Yx)=0,\ (XX)=k^2,\ (YY)=k^2.$$

Then as the lines of curvature are parametric,

$$(XY)=0,$$

and (X), (Y), (x) will be the vertices of a moving self-conjugate triangle respectively.

Again, let $\frac{\pi}{2}-\omega$ be the angle between the parametric curve $v=$ const. and the plane of the circle, and (z) the coordinates of the point of intersection of the line joining the points (X) and (Y) with the plane of the circle, then

$$(9) \qquad z_i = X_i \cos\omega + Y_i \sin\omega,\ (zz)=k^2.$$

$$(i=0, 1, 2, 3).$$

But the point (z) will be the absolute pole of the tangent to the

[1] See Coolidge, Non-Euclidean Geometry, p. 204.

circle at the point (x), the coordinates of the center of the circle will be given by

$$(10) \qquad a_i = x_i \cos\frac{R}{k} + (X_i \cos\omega + Y_i \sin\omega)\sin\frac{R}{k},$$

$$(i=0, 1, 2, 3)$$

Next, let (Z) be the absolute pole of the tangent plane to the surface S at the point (x), the point (Z) will lie on the plane of the circle and the line joining the points (x) and (Z) will be the tangent to the circle at the point (x), and

$$(xZ)=0.$$

Hence, we can put

$$(11) \qquad \xi_i = (X_i \cos\omega + Y_i \sin\omega)\cos\frac{R}{k} - x_i \sin\frac{R}{k},$$

$$\eta_i = Z_i,$$

$$(i=0, 1, 2, 3).$$

But

$$(12) \quad \begin{cases} X_i = \dfrac{k}{\sqrt{E}}\dfrac{\partial x_i}{\partial u}, \quad Y_i = \dfrac{k}{\sqrt{G}}\dfrac{\partial x_i}{\partial v}, \\ \dfrac{\partial Z_i}{\partial u} = \dfrac{1}{\tan\dfrac{r_2}{k}}\dfrac{\partial x_i}{\partial u} = \dfrac{\sqrt{E}}{k\tan\dfrac{r_2}{k}}X_i, \\ \dfrac{\partial Z_i}{\partial v} = \dfrac{\sqrt{G}}{k\tan\dfrac{r_1}{k}}Y_i, \qquad (i=0, 1, 2, 3) \end{cases}$$

where $\dfrac{1}{k\tan\dfrac{r_1}{k}}$ and $\dfrac{1}{k\tan\dfrac{r_2}{k}}$ are the curvatures of the normal sections through the tangents to the lines of curvature $u=$const. and $v=$const. respectively, and

$$DZ_i=\frac{1}{EG}\begin{vmatrix} x_i & k^2 & 0 & 0 \\ \frac{\partial x_i}{\partial u} & 0 & E & 0 \\ \frac{\partial x_i}{\partial v} & 0 & 0 & G \\ \frac{\partial^2 x_i}{\partial u^2} & -E\frac{1}{2}\frac{\partial E}{\partial u} & -\frac{1}{2}\frac{\partial E}{\partial v} & \end{vmatrix}, \quad (i=0, 1, 2, 3)$$

we have

$$\frac{\partial^2 x_i}{\partial u^2}=\frac{1}{2E}\frac{\partial E}{\partial u}\frac{\partial x_i}{\partial u}-\frac{1}{2G}\frac{\partial E}{\partial v}\frac{\partial x_i}{\partial v}-\frac{E}{k^2}x_i-\frac{DZ_i}{k^2},$$

$$(i=0, 1, 2, 3).$$

and similarly

$$\frac{\partial^2 x_i}{\partial u\partial v}=\frac{1}{2E}\frac{\partial E}{\partial v}\frac{\partial x_i}{\partial u}+\frac{1}{2G}\frac{\partial G}{\partial u}\frac{\partial x_i}{\partial v}, \tag{13}$$

$$\frac{\partial^2 x_i}{\partial v^2}=-\frac{1}{2E}\frac{\partial G}{\partial u}\frac{\partial x_i}{\partial u}+\frac{1}{2G}\frac{\partial G}{\partial v}\frac{\partial x_i}{\partial v}-\frac{G}{k^2}x_i-\frac{D''Z_i}{k^2},$$

$$(i=0, 1, 2, 3).$$

Hence we have

$$\frac{\partial X_i}{\partial u}=-\frac{k}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}Y_i-\frac{\sqrt{E}}{k}x_i-\frac{\sqrt{E}}{k\tan\frac{r_2}{k}}Z_i,$$

$$\frac{\partial Y_i}{\partial u}=\frac{k}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}X_i, \tag{14}$$

$$\frac{\partial X_i}{\partial v}=\frac{k}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}Y_i,$$

$$\frac{\partial Y_i}{\partial v}=-\frac{k}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}X_i-\frac{\sqrt{G}}{k}x_i-\frac{\sqrt{G}}{k\tan\frac{r_1}{k}}Z_i,$$

$$(i=0, 1, 2, 3).$$

From the results above we have

$$\left(\xi\frac{\partial a}{\partial u}\right)=k\left\{\sqrt{E}\cos\omega+\frac{\partial R}{\partial u}\right\},$$

$$\left(\xi\frac{\partial a}{\partial v}\right)=k\left\{\sqrt{G}\sin\omega+\frac{\partial R}{\partial v}\right\},$$

$$\left(\eta\frac{\partial a}{\partial u}\right)=-k\sin\frac{R}{k}\frac{\sqrt{E}\cos\omega}{\tan\frac{r_2}{k}}, \tag{15}$$

$$\left(\eta\frac{\partial a}{\partial v}\right)=-k\sin\frac{R}{k}\frac{\sqrt{G}\sin\omega}{\tan\frac{r_1}{k}},$$

$$\left(\eta\frac{\partial \xi}{\partial u}\right)=-k\cos\frac{R}{k}\frac{\sqrt{E}\cos\omega}{\tan\frac{r_2}{k}},$$

$$\left(\eta\frac{\partial \xi}{\partial v}\right)=-k\cos\frac{R}{k}\frac{\sqrt{G}\sin\omega}{\tan\frac{r_1}{k}},$$

Hence, the differential equation (5) will be

$$\begin{aligned}kd\theta=&\left[\left\{\frac{\sqrt{E}\cos\omega}{R'}+\frac{1}{R'(1+R'^2)}\frac{\partial R}{\partial u}\right\}\sin\theta\right.\\&\left.+\frac{\sqrt{E}\cos\omega}{\tan\frac{r_2}{k}}\frac{1}{\sqrt{1+R'^2}}(1+\cos\theta)\right]du\\&+\left[\left\{\frac{\sqrt{G}\sin\omega}{R'}+\frac{1}{R'(1+R'^2)}\frac{\partial R}{\partial v}\right\}\sin\theta\right.\\&\left.+\frac{\sqrt{G}\sin\omega}{\tan\frac{r_1}{k}}\frac{1}{\sqrt{1+R'^2}}(1+\cos\theta)\right]dv,\end{aligned} \tag{16}$$

where

$$R'=\tan\frac{R}{k}.$$

Or

$$k\tan\frac{R}{k}d\theta=\left[\left\{\sqrt{E}\cos\omega+\frac{\partial R}{\partial u}\right\}\sin\theta\right.$$

$$\left.+\frac{\sqrt{E}\cos\omega}{\tan\frac{r_2}{k}}\sin\frac{R}{k}(1+\cos\theta)\right]du \tag{17}$$

$$+\left[\left\{\sqrt{G}\sin\omega+\frac{\partial R}{\partial v}\right\}\sin\theta+\frac{\sqrt{G}\sin\omega}{\tan\frac{r_1}{k}}\sin\frac{R}{k}(1+\cos\theta)\right]dv.$$

Therefore, as remarked before, if there are three surfaces orthogonal to a congruence of circles, the congruence form a cyclical system.

The conditions of integrability will be reduced to the following two equations:

$$\frac{\partial}{\partial u}\left(\frac{\sqrt{G}\sin\omega}{\tan\frac{R}{k}}\right)-\frac{\partial}{\partial v}\left(\frac{\sqrt{E}\cos\omega}{\tan\frac{R}{k}}\right)=0,$$

$$\frac{\partial}{\partial u}\left(\frac{1}{\tan\frac{r_1}{k}}\frac{\sqrt{G}\sin\omega}{\tan\frac{R}{k}}\right)-\frac{\partial}{\partial v}\left(\frac{1}{\tan\frac{r_2}{k}}\frac{\sqrt{E}\cos\omega}{\tan\frac{R}{k}}\right) \tag{18}$$

$$+\frac{\sqrt{EG}\sin\omega\cos\omega}{k\tan^2\frac{R}{k}}\left(\frac{1}{\tan\frac{r_2}{k}}-\frac{1}{\tan\frac{r_1}{k}}\right)=0.$$

Hence, if the surface S be a sphere or a plane, the second equation of the conditions will be reduced to the first. And we have incidentally the following theorem:

Theorem. A congrnence of circles orthogonal to a sphere (or a plane) and to any other surface constitutes a cyclical system.

Next, we assume that the surface S is not the sphere, then from the relation

$$\frac{\partial}{\partial u}\left(\frac{\sqrt{G}}{\tan\frac{r_1}{k}}\right)=\frac{1}{\tan\frac{r_2}{k}}\frac{\partial\sqrt{G}}{\partial u},$$

$$\frac{\partial}{\partial v}\left(\frac{\sqrt{E}}{\tan\frac{r_2}{k}}\right)=\frac{1}{\tan\frac{r_1}{k}}\frac{\partial\sqrt{E}}{\partial v}, \tag{19}$$

[1] These relations can be obtained from the equation of Gauss (13) easily.

and the conditions (18), we have

$$k\frac{\partial \tan\frac{R}{k}}{\partial u}=k\tan\frac{R}{k}\cot\omega\left(\frac{\partial\omega}{\partial u}-\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}\right)-\sqrt{E}\cos\omega,$$

(20)

$$k\frac{\partial \tan\frac{R}{k}}{\partial v}=-k\tan\frac{R}{k}\tan\omega\left(\frac{\partial\omega}{\partial v}+\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}\right)-\sqrt{G}\sin\omega.$$

From the first condition of (18) the expression

$$\frac{\sqrt{E}\cos\omega}{\tan\frac{R}{k}}du+\frac{\sqrt{G}\sin\omega}{\tan\frac{R}{k}}dv \tag{21}$$

is the total differential. If we take the unknown complementary function ψ, then we can put

$$\frac{\sqrt{E}\cos\omega}{\tan\frac{R}{k}}=-\frac{k}{\psi}\frac{\partial\psi}{\partial u},\quad \frac{\sqrt{G}\sin\omega}{\tan\frac{R}{k}}=-\frac{k}{\psi}\frac{\partial\psi}{\partial v}, \tag{22}$$

and the second condition will be reduced to the following Laplace equation by the aid of the relation (19):

$$\frac{\partial^2\psi}{\partial u\partial v}=\frac{\partial\log\sqrt{E}}{\partial v}\frac{\partial\psi}{\partial u}+\frac{\partial\log\sqrt{G}}{\partial u}\frac{\partial\psi}{\partial v}. \tag{23}$$

Conversely, if ψ be a solution of this differential equation, then the corresponding cyclical system of circles can be given by the following equations:

$$\frac{1}{\tan^2\frac{R}{k}}=\frac{k^2}{E}\left(\frac{\partial\log\psi}{\partial u}\right)^2+\frac{k^2}{G}\left(\frac{\partial\log\psi}{\partial v}\right)^2,$$

(24)

$$\cos\omega=-\frac{k\tan\frac{R}{k}}{\sqrt{E}}\frac{\partial\log\psi}{\partial u},$$

$$\sin\omega=-\frac{k\tan\frac{R}{k}}{\sqrt{G}}\frac{\partial\log\psi}{\partial v}.$$

Let

$$\psi=(ax)+C$$

where (a) is the coordinate of a fixed point and C constant, then

$$\frac{\partial^2\psi}{\partial u\partial v}=\left(a\frac{\partial^2 x}{\partial u\partial v}\right)$$

$$=\frac{\partial\log\sqrt{E}}{\partial v}\left(a\frac{\partial x}{\partial u}\right)+\frac{\partial\log\sqrt{G}}{\partial u}\left(a\frac{\partial x}{\partial v}\right).$$

Hence

$$\psi=(ax)+C$$

is a particular solution of the Laplace equation (23) and the corresponding cyclical system of circles is orthogonal to the surface S and a sphere whose equation is

$$(ax)^2=C'.$$

This is the fact which we discussed before. When $C'=0$, the sphere will be a plane.

§ 7. The Theorem of Ribaucour.

Now, we consider a family of surfaces of a triply orthogonal system

$$\begin{aligned} &x_i=f_i(u,\ v),\\ &x_i=\phi_i(v,\ w),\\ &x_i=\psi_i(w,\ u),\quad (i=0,\ 1,\ 2,\ 3), \end{aligned}\tag{25}$$

whose lines of curvature are parametrie curves, and the system of the osculating circles of the parametric curves w of an orthogonal system at their point of intersection with a surface $w=$const. $(x_i=f_i(u,\ v))$.

Then the geodesic curvature $\frac{1}{\rho_g}$ of the osculating circles on the surface $w=$const. is given by

$$\frac{1}{\rho_g}=\frac{\sin\omega}{k\tan\frac{R}{k}}$$

(26)

$$=-\frac{1}{\sqrt{GH}}\frac{\partial\sqrt{H}}{\partial v},$$

and

(27) $$\frac{\cos\omega}{k\tan\frac{R}{k}}=-\frac{1}{\sqrt{EH}}\frac{\partial\sqrt{H}}{\partial u},$$

where

$$E=\left(\frac{\partial x}{\partial u}\frac{\partial x}{\partial u}\right),\quad G=\left(\frac{\partial x}{\partial v}\frac{\partial x}{\partial v}\right),\quad H=\left(\frac{\partial x}{\partial w}\frac{\partial x}{\partial w}\right).$$

The coordinates of the pole of the tangent plane to the surface $w=$const. at that point are

$$y_i=\frac{1}{\sqrt{EG}}\frac{\partial}{\partial r_i}\left|r\ x\ \frac{\partial x}{\partial u}\ \frac{\partial x}{\partial v}\right|,\quad (i=0,\ 1,\ 2,\ 3),$$

$$(yy)=k^2.$$

But since

$$\left(x\frac{\partial x}{\partial w}\right)=\left(\frac{\partial x}{\partial u}\ \frac{\partial x}{\partial w}\right)=\left(\frac{\partial x}{\partial v}\ \frac{\partial x}{\partial w}\right)=0,$$

we get

$$y_i=\lambda\frac{\partial x_i}{\partial w},\quad (i=0,\ 1,\ 2,\ 3)$$

where λ is a parameter.

And therefore, we have

(28) $$y_i=\frac{k}{\sqrt{H}}\frac{\partial x_i}{\partial w},\quad (i=0,\ 1,\ 2,\ 3)$$

and

$$D=\left(y\frac{\partial^2 x}{\partial u^2}\right)=-k\frac{\sqrt{E}}{\sqrt{H}}\frac{\partial\sqrt{E}}{\partial w},$$

---

1 See Coolidge, Non-Euclidean Geometry, p. 188, 208.

(29)
$$D' = \left(y\frac{\partial x}{\partial u \partial v}\right) = 0,$$

$$D'' = \left(y\frac{\partial^2 x}{\partial v^2}\right) = -k\frac{\sqrt{G}}{\sqrt{H}}\frac{\partial \sqrt{G}}{\partial w}.$$

Hence

$$\frac{1}{k\tan\frac{r_1}{k}} = -\frac{1}{\sqrt{GH}}\frac{\partial \sqrt{G}}{\partial w},$$

(30)
$$\frac{1}{k\tan\frac{r_2}{k}} = -\frac{1}{\sqrt{EH}}\frac{\partial \sqrt{E}}{\partial w}.$$

And the equations of Lamé[1] will be

$$\frac{\partial^2\sqrt{E}}{\partial v \partial w} = \frac{1}{\sqrt{G}}\frac{\partial\sqrt{G}}{\partial w}\frac{\partial\sqrt{E}}{\partial v} + \frac{1}{\sqrt{H}}\frac{\partial\sqrt{H}}{\partial v}\frac{\partial\sqrt{E}}{\partial w},$$

(31)
$$\frac{\partial^2\sqrt{G}}{\partial w \partial u} = \frac{1}{\sqrt{H}}\frac{\partial\sqrt{H}}{\partial u}\frac{\partial\sqrt{G}}{\partial w} + \frac{1}{\sqrt{E}}\frac{\partial\sqrt{E}}{\partial w}\frac{\partial\sqrt{H}}{\partial u},$$

$$\frac{\partial^2\sqrt{H}}{\partial u \partial v} = \frac{1}{\sqrt{E}}\frac{\partial\sqrt{E}}{\partial v}\frac{\partial\sqrt{H}}{\partial u} + \frac{1}{\sqrt{G}}\frac{\partial\sqrt{G}}{\partial u}\frac{\partial\sqrt{H}}{\partial v}.$$

$$\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}\right) + \frac{1}{H}\frac{\partial\sqrt{E}}{\partial w}\frac{\partial\sqrt{G}}{\partial w} = 0,$$

(32)
$$\frac{\partial}{\partial v}\left(\frac{1}{\sqrt{G}}\frac{\partial\sqrt{H}}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{1}{\sqrt{H}}\frac{\partial\sqrt{G}}{\partial w}\right) + \frac{1}{E}\frac{\partial\sqrt{H}}{\partial u}\frac{\partial\sqrt{G}}{\partial u} = 0,$$

$$\frac{\partial}{\partial w}\left(\frac{1}{\sqrt{H}}\frac{\partial\sqrt{E}}{\partial w}\right) + \frac{1}{\partial u}\left(\frac{1}{\sqrt{G}}\frac{\partial\sqrt{H}}{\partial u}\right) + \frac{1}{G}\frac{\partial\sqrt{H}}{\partial v}\frac{\partial\sqrt{E}}{\partial v} = 0.$$

When these values which can be got from (26), (27) and (30) are substituted in the equation (18), the first vanishes identically, likewise, the second, in consequence of the equations (31) and (32). Hence we get the following theorem of Ribaucour :[2]

---

1 See Lamé, Leçons sur les coordonnées curvilignes et leurs diverses applications, p. 73–79, 1859.

2 See Ribaucour, Comptes Rendus, Vol. LXX, p. 330–333, 1870.

Theorem. Given a family of surfaces of a triply orthogonal system and their orthogonal trajectories; the osculating circles to the latter at their point of meeting with any surface of the family form a cyclical system.

§ 8. A Sphere and a Cyclical System.

Let a sphere

$$(ax)^2=(aa)(xx)\cos^2\frac{R'}{k}$$

be orthogonal to a cyclical system of circles, and take the lines of curvature on the sphere (i. e. a system of great circles which are perpendicular to each other) as the parametric curves. Then the coordinates of the center of the circles of the cyclical system are given by (see § 6, (10))

$$a_i=x_i\cos\frac{R}{k}+(X_i\cos\omega+Y_i\sin\omega)\sin\frac{R}{k}$$

$$(i=0,\ 1,\ 2,\ 3).$$

As a tangent plane to a sphere is perpendicular to the diameter through the point of contact, the coordinates of the center of the sphere will be given by

$$a_i=x_i\cos\frac{R'}{k}+Z_i\sin\frac{R'}{k},\quad (i=0,\ 1,\ 2,\ 3)$$

where (x) is the coordinates of the point of contact, and (Z) that of the absolute pole of the tangent plane.

Hence, we get

$$(aa)=k^2\cos\frac{R}{k}\cos\frac{R'}{k},$$

and since the point (Z) i. e. the pole of the tangent plane to the sphere, lies on the plane of the circle, the center of the sphere (a) must lie on the planes of the circles.

From these results, we derive the following theorem:

Theorem. If a sphere be orthogonal to a cyclical system, then the planes of the circles of the system pass through the center of the sphere

and the sphere is orthogonal to every main sphere of the circles of the system.

Conversely, we can prove the following theorem easily :

Theorem. If the planes of the circles of a cyclical system pass through a fixed point, then the circles are cut orthogonally be a sphere with its center at that point.

These properties of the circles and the spheres have been discussed already by Prof. T. Nishiuchi.[1]

In conclusion the author wishes to express his sincere thanks to Prof. T. Nishiuchi for his kind guidance and encouraging remarks.

---

1 See T. Nishiuchi, 'Oriented Circles in Non-Euclidean Space,' loc. cit.

Electrolytic Reactions of Naphthalene and its Derivatives. Part II. Electrolytic Oxidation of α-Naphthol.

By

Kashichi Ono.

(Received November 22, 1921).

In my previous investigation[1] I qualitatively confirmed the presence of α-naphthol as the intermediate product of the electrolytic oxidation of naphthalene. I have now undertaken to discover the progress of the electrolytic oxidation of α-naphthol, the result of the investigation being reported in the present paper.

According to my experiments, the electrolytic oxidation of α-naphthol takes place only in an acid solution; in an alkaline solution (50 c.c. of 20% sodium hydroxide, 100 c.c. of acetone and 3 grams α-naphthol with platinum electrodes) the formation of α-naphthoquinone could not be observed. As the anode material, platinum, lead-peroxide, nickel and lead were used, of which lead-peroxide gave the best result (Table I). This is due to the higher overvoltage of lead-peroxide on the one hand and its catalytic action on the other. The suitable concentration of sulphuric acid was 10–30%, and especially was the 10% solution found the best (Table 2). The current yield of the first oxidation product was observed to become the smaller the greater the current density, the maximum yield having been attained at about 1 amp./100 sq. cm. (Table 3). The elevation of the electrolytic temperature does not necessarily favour the yield (Table 4). At the beginning of this investigation it was expected that α-naphthoquinone

[1] These memoirs, **5**, 111–130, (1921).

would be produced as the chief product as in the case of the electrolytic oxidation of naphthalene, but it was soon found that a beautifully-coloured red compound was first produced, which passed into *α*-naphthoquinone on further oxidation. I have confirmed this red substance to be the compound of *α*-naphthol and *α*-naphthoquinone. As it is evident from table 5, up to about 4 times of the theoretical electric quantity, the increase of the current quantity and of the production of the red compound run parallel. When more current is passed the yield of the red compound decreases and that of the *α*-naphthoquinone proportionally increases.

It is already known that *α*-naphthol gives *αα*-binaphthol by the action of ferric chloride,[1] *α*-naphthoquinone and phthalic acid by chromic acid[2] and phthalic acid or phthalonic acid by nitric acid[3] or potassium permanganate[4].

Now by electrolytic oxidation, besides the red compound above mentioned, *α*-naphthoquinone and phthalic acid, a dark violet substance was always produced. This dark violet substance easily dissolves in boiling alkali, and is precipitated again by acid. It gives a dark blue colouration with ferric chloride. From these facts it may be inferred that the dark violet substance contains the phenolic group in its molecule.

On fusing it with potash it gave a small quantity of phthalic acid, and a larger quantity of phthalic acid and a very small quantity of phthalonic acid were obtained when oxidized with potassium permanganate. By distilling it with zinc dust I have obtained a distillate which dissolves in benzene with an orange yellow colour showing a yellowish green fluorescence. I could also isolate naphthalene and *αα*-binaphthyl as their picrates from its distillate. From the fact that the dark violet substance gives *αα*-binaphthyl by the zinc dust distillation and possesses the phenolic property, there is little room for doubting that *αα*-binaphthol is produced as a step in the oxidation. *αα*-Binaphthol also gives *α*-naphthoquinone together with some phthalic acid on further oxidation as may be seen in the experimental parts of this report.

---

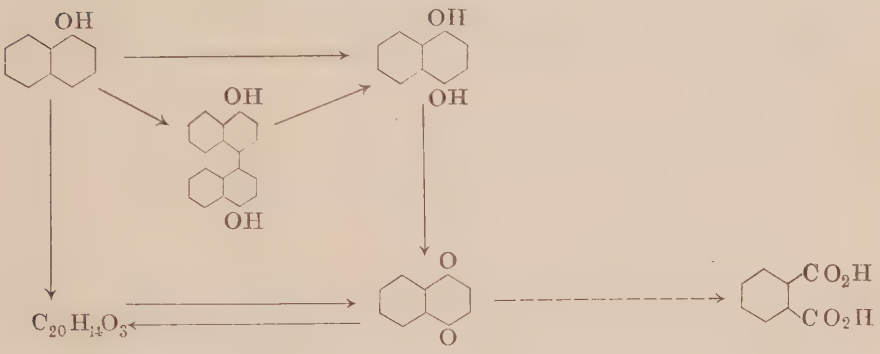
1 Dianin, J. d. russ. phys. chem. Ges., **6**, 183 (1874); Beilstein's Handb. org. Chem., 1896, **11**, 1004.

2 O. Miller, Ber., **14**, 1600 (1881); K. Ono, unpublished.

3 Thorpe, A Dictionary of Applied Chem., 1912, III, 615.

4 R. Henriques, Ber., **21**, 1608 (1888).

Judging from results so far obtained, I venture to assume that the progress of reaction of the electrolytic oxidation of *α*-naphthol may be represented by the following scheme:



It is here to be mentioned that the red compound was also formed in the cathode room just as I saw it by the oxidation of naphthalene.

EXPERIMENTAL PART.

α-Naphthol recrystallized from petroleum ether and melting at 96° was used.

(1) Preliminary Experiment.

There is apparently no record in chemical literature of the electrolytic oxidation of *α*-naphthol. In order to ascertain what sort of substance is produced by its electrolytic oxidation, the following experiments were performed.

(1) The anode liquid consisting of 100 c.c. of sulphuric acid (10%), 50 c.c. of acetone and 6 grams of *α*-naphthol was introduced into a beaker. A porous cell, successively washed with 1% alkali, water, 5% hydrochloric acid, and water, was placed in it as the cathode compartment. Sulphuric acid (20%) was taken as the cathode liquid, and a sheet of lead covered with lead-peroxide, about 10.0 × 8.0 cm. served as the cathode. The beaker was cooled with water to keep the temperature of the bath always at about 20°. The

electrolysis was continued for nearly 4.4 ampère hours with a current of 0.8 ampère.

After electrolysis the anode liquid was extracted with ether. The solvent was then evaporated, and the residue subjected to steam distillation. The red crystalline distillate after having been dried was crystallized three times from petroleum ether, and deep red needle crystals, melting at 96–97°, were obtained. It was analyzed by K. H. Meyer's method.

0.1187 grams substance required 7.9 c.c. of N/10 sodium thiosulphate.

	Found	Cal. ($C_{10}H_6O_2.C_{10}H_8O$)
α-Naphthoquinone	52.60%	52.31%

From this analytical result it may be inferred that the deep red substance consists of a molecule of *α*-naphthol and of *α*-naphthoquinone. As it dissociates in the organic solvent its molecular weight could not be determined in the usual way. Its solubilities were found as follows:

Solvent	Methyl alcohol	Ethyl alcohol	Amyl alcohol	Acetone	Ether	Chloroform	Carbon bisulphide	Petroleum ether	Acetic acid	Benzene	Toluene	Water
Solubility	soluble	,,	,,	,,	,,	,,	,,	cold—difficult hot—a little	soluble	,,	,,	cold—very difficult hot—difficult
Colour in solution	yellow	,,	,,	,,	,,	,,	,,	,,	,,	,,	,,	,,

To separate two components of the red substance from one another the following experiment was performed.

One gram of the red substance was dissolved in a small quantity of hot benzene and a saturated hot benzene solution of picric acid (0.77 gram) was added. On cooling the orange red picrate formed was filtered, washed with a small quantity of carbon bisulphide and dried. Thus about 1.22 grams of the picrate was obtained. Theoretically 1 gram of the substance having composition $C_{10}H_6O_2.C_{10}H_8O$ should give 1.235 grams of naphthol picrate.

The picrate after having been recrystallized twice from alcohol, was found to melt at 189–190° and was proved to be identical with *α*-naphthol picrate by a direct comparison with a pure specimen.

The filtrate from the picrate was distilled, and the residue was

subjected to steam distillation. A nearly quantitative yield of α-naphthoquinone was obtained.

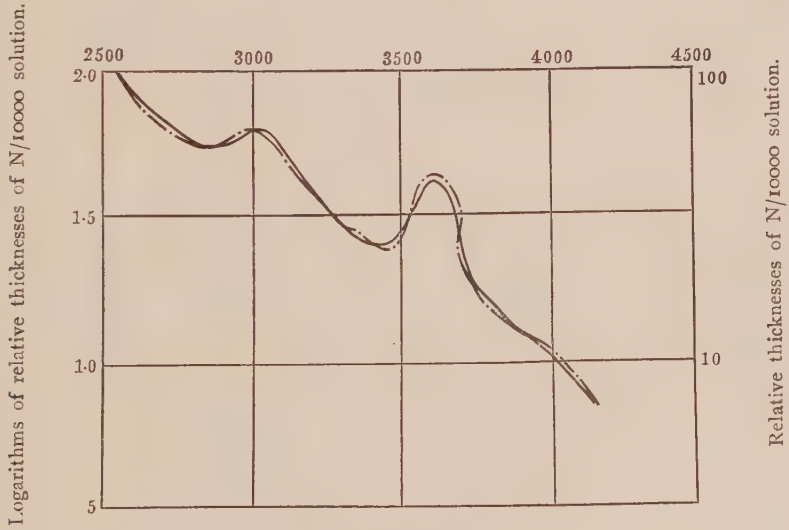
With the object of comparing the red substance electrochemically obtained with α-naphthol-α-naphthoquinone chemically formed I prepared it by mixing the two components in a petroleum ether solution (K. H. Meyer[1]). Both are found nearly the same; the only difference observed was their colour, that electrochemically prepared having a deeper tint.

The red substance was analyzed with the following results:

0.1865 grs. substance gave 0.5416 gr. CO_2 and 0.0758 gr. H_2O

	Found	Cal. ($C_{20}H_{14}O_3$)
carbon	79.20%	79.44%
hydrogen	4.54	4.63

For the further confirmation of the identity of the two preparations, spectrographic experiments were conducted according to the well known method of Hartly. For this purpose, their N/10000 solutions were preparcd with 20% (vol.) alcohol and a Hilger quartz spectrograph was employed.



[1] Ber., **42**, 1153 (1909).

The absorption curve was plotted from the photograph as usual; according to Hartly-Baly's method.[1] As is seen in the figure the curves showing the absorption spectra of the two preparations almost agree with each other, making their identity doubtless.

The anode liquid from which α-naphthol-α-naphthoquinone was distilled off was filtered while hot, and the filtrate was evaporated to a small volume. After dissolving it in a sodium hydroxide solution, the carbon dioxide was passed into the alkaline solution until the phenolic precipitate was no longer formed, and filtered. On acidifying the filtrate, a small quantity of the precipitate separated out, which yielded the long white needles melting at 128° by sublimation. The substance was identified by the fluorescence reaction as phthalic acid anhydride.

(2) In the second experiment the electrolysis was continued for nearly 15.2 ampère hours with a current of 1.6 ampère under the same condition as before. The yellow crystals identified as α-naphthoquinone by the colour reactions with alcoholic phenylhydrazine[2] and with alcoholic aniline[3] was obtained by distilling the anode liquid with steam. It was further characterized as an oxime by heating with hydroxylamine hydrochloride by the method of H. Goldschmidt and H. Schmidt.[4]

Phthalic acid was detected in the residue of the steam distillation.

(II) Quantitative Investigation.

Having now confirmed the fact that α-naphthol can be oxidized into α-naphthol-α-naphthoquinone, α-naphthoquinone and phthalic acid, I next attempted to examine the oxidation reaction quantitatively under various conditions. In all experiments I took care to work at a definite current concentration and temperature.

i. Influence of the anode material.

Anode solution: 3.0 gr. α-naphthol
100 c.c. acetone
50 c.c. H_2SO_4(10%)

---

1 Baly and Desch, J. Chem. Soc., **85**, 1029 (1904).
2 J. Zinck and H. Bindewald, Ber., **17**, 3026 (1884).
3 R. Plimpton, J. Chem. Soc., **37**, 635 (1880).
4 Ber., **17**, 2064 (1884).

Cathode solution : H_2SO_4(10%)
Temperature : 18–23°.

TABLE I.

Anode		Cathode	Current	Voltage	Current quantity	α-naphthol-α-naphthoquinone	α-naphthoquinone	Current yield of α-naphthol-α-naphthoquinone	Material yield of α-naphthol-α-naphthoquinone
Material	Surface area		strength						
	sq.cm.		amp.	volt	amp.hr.	gr.	gr.	%	%
Platinum	40.0	Spiral platinum	0.8	3.1–3.2	2.24	0.4307	0.2253	6.82	13.70
Lead peroxide	80.0	Lead	1.6	3.2–3.3	2.24	1.1243	0.5881	17.80	35.75
Nickel	90.0	Nickel	1.8	3.4–3.6	2.25	0.1510	0.0790	2.37	4.80
Lead	80.0	Spiral platinum	1.6	2.8–3.2	2.24	0.3309	0.1731	5.24	10.53

The yield of *α*-naphthol-*α*-naphthoquinone was calculated from the value of *α*-naphthoquinone found analytically, according to the formula ($C_{10}H_6O_2.C_{10}H_8O$). The current yield was calculated on the assumption that the oxidation takes place according to the following equation :

$$2C_{10}H_8O + 2O = C_{10}H_6O_2.C_{10}H_8O + H_2O$$

ii. Influence of the concentration of sulphuric acid.

Anode solution : 3.0 gr. *α*-naphthol
100 c.c. acetone
50 c.c. H_2SO_4
Anode : Lead peroxide
Cathode solution : H_2SO_4 (The same concentration as in the anode solution)
Cathode : Lead
Current strength : 1.6 ampère
Current quantity : 2.24 ampère hours
Temperature : 18–23°

TABLE II.

Concentration of sulphuric acid	Voltage	α-naphthol-α-naphthoquinone	α-naphtho-quinone	Current yield of α-naphthol-α-naphtho-quinone	Material yield of α-naphthol-α-naphtho-quinone
%	volt	gr.	gr.	%	%
10	3.2–3.3	1.1243	0.5881	17.80	35.75
20	3.5–3.8	0.8386	0.4387	13.28	26.67
30	3.7–4.2	0.7692	0.4023	12.18	24.24
40	4.0–4.4	0.6618	0.3462	10.47	21.05
50	4.3–4.5	0.4624	0.2419	7.32	14.71

iii. Influence of current density.

Anode solution: 3.0 gr. *α*-naphthol
100 c.c. acetone
50 c.c. H_2SO_4(10%)
Anode: Lead peroxide
Cathode: solution: H_2SO_4(10%)
Cathode Lead
Current quantity: 2.24 ampère hours
Temperature: 18–23°.

TABLE III.

Current density	Voltage	α-naphthol-α-naphthoquinone	α-naphtho-quinone.	Current yield of α-naphthol-α-naphthoquinone	Material yield of α-naphthol-α-naphthoquinone
amp./100cm.^{2}	volt	gr.	gr.	%	%
4	4.0–4.3	0.7601	0.3976	12.04	24.18
3	3.5–3.8	0.8356	0.4371	13.23	26.58
2	3.2–3.3	1.1243	0.5881	17.80	35.75
1	2.9–3.1	1.4734	0.7707	23.33	46.87
0.5	2.2–2.5	1.0864	0.5683	17.20	34.56

iv. Influence of electrolytic temperature.

Anode solution: 3.0 gr. *α*-naphthol
100 c.c. acetone
50 c.c. H_2SO_4(10%)

Anode : Lead peroxide
Cathode solution : H_2SO_4(10%)
Cathode : Lead
Current strength : 1.6 ampère
Current quantity : 2.24 ampère hours

TABLE IV.

Temperature	Voltage	α-naphthol-α-naphthoquinone	α-naphthoquinone	Current yield of α-naphthol-α-naphthoquinone	Material yield of α-naphthol-α-naphthoquinone
°C	volt	gr.	gr.	%	%
18–23	3.2–3.3	1.1243	0.5881	17.80	35.75
40–53	3.4–3.6	0.7552	0.3952	11.96	24.01

v. Influence of the electric quantity.

Anode solution : 3.0 gr. α-naphthol
100 c.c. acetone
50 c.c. H_2SO_4(10%)
Anode : Lead peroxide
Cathode solution : H_2SO_4(10%)
Cathode : Lead
Current density : 0.02 ampère /sq. cm.
Temperature : 18–23°.

TABLE V.

Current quantity	α-naphthol-α-naphthoquinone	α-naphthoquinone	α-naphthoquinone current quantity	Current yield of α-naphthol-α-naphthoquinone	Material yield of α-naphthol-α-naphthoquinone	Current yield of α-naphthoquinone	Material yield of α-naphthoquinone
amp.hr.	gr.	gr.	gr./amp. hr.	%	%	%	%
0.56	0.4231	0.2213	0.395	26.79	13.46	—	—
1.12	0.7404	0.3873	0.345	23.44	23.55	—	—
2.24	1.1243	0.5881	0.262	17.80	35.75	—	—
4.48	0.6738	0.7935	0.177	5.33	21.43	6.70	13.36
6.72	0.0893	0.9138	0.135	0.47	2.84	9.20	27.54
8.96	0.0181	0.9477	0.105	0.07	0.57	7.12	29.03
11.20	—	0.8292	0.070	—	—	5.01	25 12

When α-naphthol-α-naphthoquinone and α-naphthoquinone coexisted in the oxidation product, it was first subjected to steam distillation and the distillation product collected in a Gooch crucible, dried and weighed (a). The total amount of α-naphthoquinone was then estimated (b). As (a-b) shows the amount of α-naphthol existing as α-naphthol-α-naphthoquinone the yield of α-naphthol-α-naphthoquinone may easily be calculated.

vi. Influence of the solvent.

In order to know the influence of the solvent on the oxidation, an acetic acid solution (10% sulphuric acid 50 c.c., glacial acetic acid 100 c.c., and α-naphthol 3.0 grs. with platinum electrodes (40 sq. cm.) 0.8 ampère, 2.24 ampère hours) was tried, and as its result only a very small quantity of α-naphthol-α-naphthoquinone and a little larger quantity of phthalic acid were obtained.

(III) Oxidation Products of α-Naphthol.

Six grams of α-naphthol were dissolved in a mixture of 150 c.c. of acetone and 100 c.c. of sulphuric acid (20%). A sheet of lead covered with lead peroxide, about 10.0×8.0 cm. served as the anode, and the electrolysis was continued for 4.8 ampère hours with a current of 1.6 ampères. The same experiment was repeated three times and the resultant product was treated as in the preliminary experiment. On distilling the product with steam the same red crystalline mass as in the preliminary experiment was obtained. The residue was divided into three parts:

Part soluble in cold water (A)
that soluble in hot water (B)
that insoluble in hot water (C).

Treatment of (A)—(A) was extracted with a large quantity of ether. The brownish red ethereal extract which did not decolorize with animal charcoal was evaporated, and the residue was subjected to fractional distillation.

(a) 250–275°C reddish oily substance, a little later solidified.
(b) 275–285°C yellowish oily substance, a little later crystallized.
(c) residue.

The distillate (a) was pressed on a tile and dark red crystals were obtained. It was recrystallized three times with petroleum ether and formed pale brown crystals, melting at 82–83°. It gave the orange yellow picrate with picric acid, melting at 177–178°.

The distillate (b) which was obtained in the form of brown crystals, was crystallized twice with petroleum ether. It melted at 96°, its picrate melting at 189–190°. They were proved to be identical with α-naphthol and its picrate by a direct comparison with the pure specimens.

After allowing the residue to stand for a few days, a small quantity of colourless needles melting at 184–185° was separated. A reddish brown colouration followed by precipitation was produced by adding a few drops of ferric chloride, but the fluorescence reaction could not be perceived.

After separating these crystals the residue was dissolved in a caustic soda solution and reprecipitated with hydrochloric acid and the precipitates were sublimed. The sublimed crystals melting at 128° were proved by the fluorescence reaction to be phthalic acid anhydride.

Treatment of (B)—(B) was dissolved in a small quantity of a sodium hydroxide solution, and carbon dioxide was passed into it until the phenolic precipitate was no longer formed, and filtered. The precipitate was treated with diluted mineral acid, dried and extracted with petroleum ether. On distilling off the solvent, the white needles separated out, which was confirmed to be α-naphthol.

The substance insoluble in petroleum ether forms a dark brown picrate, but its further study is not yet concluded.

Treatment of (C)—(C) quite insoluble in boiling water, but soluble in acetone and ether, etc., was easily dissolved by boiling it with alkali from which it was again precipitated by acids. It gave a dark blue colouration with ferric chloride. From these facts it might be inferred that this substance contains the phenolic group. To separate a trace of α-naphthol it was washed with a small quantity of hot benzene and then treated as follows:

(I) Potash Fusion.

Three grams of this substance were mixed with 5 c.c. of water and 3 grams of potassium hydroxide and heated for 3 hours at 200—250° in the oil bath. After cooling, the fused mass was dissolved in a little quantity of boiling water, acidified with hydrochloric acid, and filtered. The precipitate was dissolved in a sodium hydroxide solution, and carbon dioxide was passed into it until the phenolic precipitate no longer came out and filtered. A small quantity of the precipitate formed by acidifying the filtrate was dried and sublimed. By

this treatment the long white needles melting at 128° was obtained and confirmed by the fluorescence reaction as phthalic acid anhydride.

(II) Oxidation With Potassium Permanganate.

Two grams of the substance were dissolved in 60 c.c. of 10% sodium hydroxide solution and the solution was warmed with a mixture of 15 grams of potassium permanganate and 250 c.c. of water for some hours with occasional shaking until the solution was decolorized. When the oxidation was completed, the product was strongly acidified with a concentrated sulphuric acid and then evaporated to a small volume. It was extracted with ether, dried and treated with animal charcoal. On distilling off ether, some pale brown crystals separated out. After drying it was dissolved in chloroform, when its greater part remained insoluble. It was dissolved in caustic soda and reprecipitated with hydrochloric acid and, after drying, purified by sublimation. 0.265 gram of a substance melting at 128° and proved to be phthalic acid anhydride by the fluorescence reaction and by a direct comparison with a pure specimen, was obtained.

The part soluble in chloroform when decolorized with animal charcoal was isolated as a pale brown crystalline substance. It was crystallized twice from hot water and dried at 110°. It melted at 144°, and proved to be phthalonic acid.

(III) Zinc Dust Distillation.

Two grams of the substance were subjected to the zinc dust distillation according to Gatterman's method. White crystals were observed to sublime on the wall of the receiver cooled with ice and common salt and at the same time yellow crystals were seen to deposit on the cooler part of the combustion tube.

The white crystals thus obtained had a strong odor of naphthalene. It was confirmed to be naphthalene by examination of its picrate.

The yellow crystals deposited on the cooler part of combustion tube was extracted with a large amount of benzene, when it gave an orange yellow solution with yellowish green fluorescence. The solution was washed with diluted caustic soda solution and evaporated to drive off benzene, when yellow crystals were obtained. It was carefully fractionated into parts soluble and less soluble in alcohol.

The portion soluble in alcohol was recrystallized with alcohol,

and its benzene solution was added to a benzene solution of picric acid. The picrate purified by recrystallisation from benzene, formed reddish brown needles melting at 145°. Judging from this melting point, it is presumably *αα*-binaphthyl picrate.[1].

The part with difficulty soluble in alcohol was very small in quantity so that its further examination was impracticable.

(IV) Oxidation of *αα*-Binaphthol.

The *αα*-binaphthol used in my experiment was prepared from *α*-naphthol by oxidizing it with ferric chloride according to Morio's[2] method, and melted at 300–301°.

One gram of *αα*-binaphthol was dissolved in a mixture of 100 c.c. of acetone and 50 c.c. of 10% sulphuric acid. A sheet of lead covered with lead peroxide, about 10.0 × 8.0 cm. served as the anode, and the electrolysis was continued for 5.6 ampère hours with a current of 1.6 ampère.

After electrolysis was over the anode liquid was extracted with ether. The solvent was then evaporated, aud the residue was subjected to steam distillation. The distillation product that came out was identified as *α*-naphthoquinone by the colour reactions. It was recrystallized from petroleum ether and was found to melt at 125°. By comparing it with a pure specimen it was proved to be *α*-naphthoquinone. Phthalic acid was detected in the residue of the steam distillation.

(V) Oxidation of *α*-Naphthol-*α*-naphthoquinone.

One gram of *α*-naphthol-*α*-naphthoquinone which was prepared by electrolysis was dissolved in a mixture of 50 c.c. of acetone and 75 c.c. of 10% sulphuric acid. A sheet of lead covered with lead peroxide, about 10.0 × 8.0 cm. served as the anode and electrolysis was continued for 2 ampère hours with a current of 0.8 ampère.

After electrolysis the anode liquid was extracted with ether. The solvent was then evaporated, and the residue was subjected to steam distillation. The distillation product that came out was identified as *α*-naphthoquinone by the colour reactions. On recrystallizing it from petroleum ether, 0.184 gram of a substance melting at 125° was

---

1 H. Walder, Ber., **15**, 2171 (1882).

2 Unpublished.

obtained and it was proved to be α-naphthoquinone. Phthalic acid was also detected in the residue of the steam distillation.

In conclusion, the author wishes to express his sincere thanks to Prof. M. Matsui for his valuable suggestions and kind instructions throughout the work, and to Prof. M. Kimura and Mr. Nakamura for their kind suggestions and advice about the spectrographic experiment and to Mr. Morio for the loan of his unpublished manuscripts.

Vol. V, No. 4, Published February 25th, 1922.

Price - - - Yen 2.20

These Memoirs are on sale at

MARUZEN COMPANY, LTD.

TOKYO, OSAKA, KYOTO, YOKOHAMA, FUKUOKA & SENDAI.

大正十一年二月二十二日印刷

大正十一年二月二十五日發行

編纂兼發行者　京都帝國大學理學部

印刷者　大久保秀次郎

東京府下北豐島郡巢鴨町三丁目十番地

印刷所　株式會社 東京築地活版製造所

東京市京橋區築地二丁目十七番地

賣捌所　丸善株式會社

東京、大阪、京都、横濱、福岡、仙臺

CONTENTS.

On the Octahydrate of Magnesium Sulphate.
By Shirow Takegami 191

On the Figures produced on Photographic Plates by Electric Discharges.
By Shinsuke Tanaka 201

On the Solutions of Partial Differential Equations of the First Order at the Singular Points. IV.
By Toshizô Matsumoto 205

Catalytic Reduction of Dextro-Camphor. I.
By Shigeru Komatsu and Bunkichi Masumoto. 225

Congruence of Circles in Non-Euclidean Space.
By Hidetoshi Kashiwagi. 233

Electrolytic Reactions of Naphthalene and its Derivatives. Part. II. Electrolytic Oxidation of α-naphthol.
By Kashichi Ono 263